Mathematics 113 — Professor K. A. Ribet

Another Midterm Exam — September 26, 2013

Morning Edition

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*. Your explanations are your only representative when your work is being graded.

The problems have equal weight.

1. Find two non-isomorphic finite groups, each with exactly three conjugacy classes. (Explain why the groups have the required property and why they're not isomorphic to each other.)

2. Suppose that G is a finite group and that N is a normal subgroup of G. Let $\pi : G \to G/N$ be defined as usual by $g \mapsto gN$.

a. If H is a subgroup of G, show that $\pi(H)$ is isomorphic to $H/(H \cap N)$.

b. Let p be a prime number. Suppose that P is a subgroup of G whose order is the largest power of p dividing the order of G. (Thus P is a p-Sylow subgroup of G if the order of G is in fact divisible by p.) Show that the order of $\pi(P)$ is the largest power of p dividing the order of G/N.

3. Let G be a group (possibly infinite), and let $Z(G)$ be the center of G. Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.

4. Let H and K be normal subgroups of the group G such that that $H \cap K$ is the trivial group. Show that $hkh^{-1}k^{-1}$ belongs both to H and to K and then prove that $hk = kh$ for all $h \in H, k \in K$.

5. Let G be a transitive permutation group acting on the finite set A. Assume that A has at least two elements. As usual, for each $a \in A$ we let G_a be the stabilizer of a in G. Recall (from HW #8) that a *block* is a non-empty subset B of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$. Recall also that G is said to be *primitive* if the only blocks are the sets of size 1 and A itself.

a. Prove that if B is a block containing the element a of A, then the subgroup

$$G_B = \{\, \sigma \in G \mid \sigma(B) = B \,\}$$

of G contains G_a.

b. Assume that the transitive group G is primitive on A. Prove that, for each $a \in A$, the subgroup G_a of G is maximal (i.e., that there are no subgroups of G containing G_a other than G_a and G).