Mathematics 113 — Professor K. A. Ribet

First Midterm Exam — September 26, 2013

Afternoon Edition

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*. Your explanations are your only representative when your work is being graded.

The problems have equal weight.

1. Suppose that G is a finite group and that $g \in G$ has order n (where n is a positive integer). Let i be an integer. Find a formula for the order of g^i and prove that your formula is correct.

2. Suppose that H is a finite group in which each non-identity element has order 2. Prove that H is abelian.

3. Let x be an element of the dihedral group D_{2n} ($n \geq 3$). Describe explicitly the set of conjugates of x (i.e., the set of elements of the form gxg^{-1}). Treat separately the cases where x is a power of r and where x is not a power of r.

4. Let σ be the 20-cycle $(1\ \ 2\ \ 3\ \ 4\ \ \cdots\ \ 17\ \ 18\ \ 19\ \ 20)$. What are the different cycle types that occur as we consider the various powers of σ? For which integers i is σ^i a 20-cycle?

5. Let p be a prime number. Find the number of invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbf{Z}/p\mathbf{Z}$. For $t \in (\mathbf{Z}/p\mathbf{Z})^*$, show that the number of such matrices with determinant t is equal to the number of such matrices with determinant 1. What is the latter number?