Mathematics 113 — Professor K. A. Ribet

Yet Another Exam — December 20, 2013

Friday Night Edition

237 Hearst Gym

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*. Your explanations are your only representative when your work is being graded.

Name: ______________________ SID: ______________________

Problem	Value	Your Score
1	6	
2	4	
3	8	
4	5	
5	6	
6	6	
7	5	
Total	40	

1. Let G be a finite group, and let N be a normal subgroup of G. Suppose that H is a subgroup of G. Prove that the index $(H : (H \cap N))$ divides the index $(G : N)$. Deduce that if H is a subgroup of A_n, then $(H : (H \cap A_n)) \leq 2$.

2. Write $(1\,2)(1\,2\,3)(1\,2\,3\,4)(1\,2\,3\,4\,5)$ as a product of disjoint cycles in S_5.

3. Suppose that G is a group of order $3825 = 3^2 \cdot 5^2 \cdot 17$.

a. Show that G has a unique subgroup N of order 17.

b. Show that the group N in part (a) is a subgroup of the *center* of G.

4. Let R be a commutative ring with identity. When n is an integer, write n_R for the element of R corresponding to n. For example, $3_R = 1 + 1 + 1$, where each "1" in the equation is the identity element of R. If n and m are relatively prime integers, show that the ideal (n_R, m_R) in R is all of R.

5. Suppose that G is a finite group of p-power order (where p is a prime number).

a. Let A be a finite G-set (i.e., a set with an action of G). Prove the congruence $|A| \equiv |A^G| \bmod p$, where A^G is the set of elements of A that are fixed by all elements of G.

b. Suppose that $N \neq \{1\}$ is a normal subgroup of G. Show that $N \cap Z(G)$ is not the trivial group.

6. Find the gcd of $11 + 7i$ and $18 + i$ in $\mathbf{Z}[i]$.

7. Let R be a commutative ring with identity. Suppose that for each $a \in R$ there is an integer $n > 1$ such that $a^n = a$. Prove that every prime ideal of R is a maximal ideal.