Mathematics 113 Professor K. A. Ribet

Yet Another Exam December 18, 2013

Morning Edition

9 Evans Hall

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences.* Your explanations are your only representative when your work is being graded.

Name: ______________________ SID: ______________________

Problem	Max Points	Your Score
1	4	
2	5	
3	7	
4	4	
5	5	
6	5	
7	5	
8	5	
Total	40	

1. Find the smallest positive integer n for which the alternating group A_n has an element of order 1000.

2. Show that every group of order 12 has a normal Sylow subgroup.

3. Let R be an integral domain.

a. Explain what it means for an element of R to be *prime* and what it means for an element of R to be *irreducible.*

b. Show that 2 is an irreducible element, but not a prime element, of the ring $\mathbf{Z}[\sqrt{-3}]$.

c. Suppose that all ideals of R are principal. If r is an irreducible element of R, show that the ideal (r) is maximal and that r is a prime element of R.

4. Let A and B be subsets of a finite group G for which $|A| + |B| > |G|$. Let g be an element of G, and let $gB^{-1} = \{\, gb^{-1} \mid b \in B \,\}$. Show that $A \cap gB^{-1} \neq \emptyset$ and conclude that $g = ab$ for some $a \in A$, $b \in B$.

5. This problem concerns $n \times n$ matrices of real numbers.

a. Suppose that M is such a matrix and that X and Y are $n \times n$ matrices with a single non-zero entry, which is equal to 1. Describe the product XMY in terms of the entries of M and the positions of the non-zero entries in X and Y.

b. Show that the ring of $n \times n$ matrices of real numbers has no two-sided ideals other than (0) and the whole ring.

6. Let C be a cyclic group of order p^n, where p is an odd prime number and n is a positive integer. Show that C has a unique automorphism of order 2.

7. Suppose that I and J are ideals of a commutative ring R with the property that the canonical map

$$R \longrightarrow R/I \times R/J$$

is surjective ("onto"). Show that I and J are comaximal in the sense that $I + J = R$.

8. Let n be a positive integer. Let R be the ring $\mathbf{C}^n$ whose elements are n-tuples of complex numbers and whose ring operations are componentwise addition and multiplication. For each i, $1 \le i \le n$, let $\pi_i : R \to \mathbf{C}$ be the ith projection $(x_1, \ldots, x_n) \mapsto x_i$.

a. Show that the kernel of π_i is a maximal ideal of R.

b. Prove that each maximal ideal of R is the kernel of π_i for some i.