Math H113 Professor K. A. Ribet

Midterm Exam February 20, 2003

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

These solutions were written quickly by Ken Ribet. Sorry if they're a little terse. They're perhaps best described as sketches of solutions.

1. (5 points) Find integers x and y that satisfy $47x + 63y = 1$.

I carried out the Euclidean algorithm and found $63 = 47+16$, $47 = 2\cdot 16+15$, $16 = 15 + 1$. *This gave* $1 = 3 \cdot 63 - 4 \cdot 47$ *on substitution.*

2. (6 points) Let N_1 and N_2 be normal subgroups of a group G. Suppose that x is an element of N_1 and that y is an element of N_2. Show that $xyx^{-1}y^{-1}$ belongs to the intersection $N_1 \cap N_2$.

Because N_2 *is normal,* xyx^{-1} *belongs to* N_2. *Therefore,* $xyx^{-1}y^{-1} = (xyx^{-1})y^{-1}$ *belongs to* N_2. *Similarly,* $xyx^{-1}y^{-1} = x(yx^{-1}y^{-1})$ *belongs to* N_1.

3. (9 points) Let H be a subgroup of a finite group G. Recall that the normalizer $N_G(H)$ of H is the group of all $g \in G$ such that $gHg^{-1} = H$. Show that the number of distinct subgroups gHg^{-1} (with $g \in G$) is the index $(G : N_G(H))$ and that this number divides $(G : H)$. If H is smaller than G, show that the union of the gHg^{-1} is smaller than G.

Let G *act by conjugation on the set of subgroups of* G. *The normalizer* $N_G(H)$ *is the stabilizer of the point* H *under this action. The set of conjugates of* H *is the orbit of* H *under this action. In general, we have a relation "*$G/G_s \xrightarrow{\sim} O(s)$*" when* s *is an element of a set on which* G *acts. The number of elements of* $O(s)$ *is then the number of elements of* G/G_s, *i.e., the index* $(G : G_s)$. *In the present case, this equality shows that*

$(G : N_G(H))$ is the number of conjugates of H, as required. Now we have shown in general that $(G : J) = \#(G)/\#(J)$ when J is a subgroup of G. We have $G \supseteq N_G(H) \supseteq H$; it follows from the general equality that we have $(G : N_G(H))(N_G(H) : H) = (G : H)$. Hence $(G : N_G(H))$ divides $(G : H)$, as was to be shown. Finally, consider the union of the various gHg^{-1}. Each set in the union has cardinality (= size) $\#(H)$. The number of different subgroups in the union is at most $(G : H)$: it's a divisor of $(G : H)$, as we have seen. Recall that $\#(G) = (G : H)\#(H)$. Hence the union can fill up all of G only if: (1) there are as many factors as possible (namely, $(G : H)$ of them) , and (2) the union is disjoint. Assume that H is a proper subgroup of G, so that $(G : H)$ is at least 2. Then if (1) holds, there are at least two different groups gHg^{-1}. It follows that the union will not be disjoint because the identity element is an element of each subgroup. Hence (2) is violated, so the union misses some element(s) of G.

4. (5 points) Consider the products $\sigma = (123)(45)(243)$, $\tau = (243)(45)(123)$ in the symmetric group $\mathbf{S}_5$. Write each as a product of disjoint cycles. Show that σ and τ are conjugate in $\mathbf{S}_5$.

When I worked this out, I got that the first product is $(1\,2\,5\,4)(3) = (1\,2\,5\,4)$ and that the second product is $(1\,4\,5\,3)$. If p is the first product and q is the second, then I find that $q = \sigma p \sigma^{-1}$ where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix}$.

5. (5 points) Let G be a group with the following property: if two elements of G commute with each other, either the elements are equal or at least one of them is the identity. Show that G has at most 2 elements.

Let g be an element of G. Clearly, g commutes with its powers. Since these powers are constrained to be either g or the identity e, g must have order 1 or 2. Hence $g^2 = e$ for all $g \in G$. This condition clearly implies that G is abelian as you saw in problem 25 of section 1.1 of the book. Hence all elements of G commute with each other! Thus there cannot be two distinct non-identity elements of G, so G must have order ≤ 2.