Math 113H

April 8, 1991

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1. *Show that all groups of order 22 are either cyclic or dihedral.* Let G have order 22. Let N be the 11-Sylow of G. Then N is a normal subgroup of G, since it has index 2. Let $H = \{h, e\}$ be a 2-Sylow subgroup of G. We have $H \cap N = (e)$ so that $N \cdot H$ has order 22 and thus must coincide with G. Hence G is a semidirect product of N by H. The action of H on N is given by a homomorphism $H \to \operatorname{Aut} N$, which is determined by the image of h in $\operatorname{Aut} N$. This image is either the identity automorphism, or else an automorphism of order 2, since h has order 2. But $\operatorname{Aut} N$ is a cyclic group of order 10, and therefore has precisely one element of order 2, namely $n \mapsto n^{-1}$. Hence there are exactly two possibilities for $H \to \operatorname{Aut} N$. The one where h maps to $n \mapsto n^{-1}$ gives the dihedral group, according to the definition of this group. The one where h maps to the identity makes G abelian, and therefore cyclic. (If G is abelian and n has order 11, then nh has order 22.)

2. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 6 & 8 & 1 & 5 & 2 \end{pmatrix}$.

a. *Write σ as a product of disjoint cycles.* The answer is $(1\,4\,6)(2\,3\,7\,5\,8)$, as almost all of you found.

b. *Find* $\operatorname{sgn} \sigma$. Both cycles have odd length, and thus sign $+1$. The product, σ, thus has sign $+1$.

c. *Calculate $\tau\sigma\tau^{-1}$, where $\tau = (1\,2\,3)(4\,5\,6)$.* You should do this by replacing each number in the cycle decomposition of σ by the image of this number under τ: $(2\,5\,4)(3\,1\,7\,6\,8)$. It's of course possible to do the calculation longhand, but this takes a lot of time.

d. *How many conjugates does σ have in S_8?* I took this off the exam and thought I'd save it for the final.

3. a. *For which n does the symmetric group S_n have an element of order 15?* The order of an element of S_n is the least common multiple of the lengths of the non-trivial disjoint cycles whose product is the element. If the order is 15, these lengths must all equal 3, 5, and 15. Further, either 15 must occur at least once or else 3 and 5 must each occur at least once. Hence the sum of the lengths of the cycles must be at least 3+5 = 8, so $n \geq 8$. Conversely, if $n \geq 8$, then we can make a cycle of order 15, e.g., the one in question 2.

b. *For which n does S_n have a subgroup of order* 15*?* Since all groups of order 15 are cyclic, the answer to this question is the same as the answer to the previous one: $n \geq 8$. Notice, for example, that S_5 has order 5!, which is divisible by 15. Yet we have shown that S_5 does not have a subgroup of order 15. This shows that the following generalization of Sylow's theorem is false: suppose d divides the order of a finite group G. Then G has a subgroup of order d.

4. *Let N be a normal subgroup of the group G. Assume that $N \neq (e)$ and that G is finite with p-power order (where p is a prime). Show that $N \cap Z \neq (e)$, where Z is the center of G.* Let G operate on N by conjugation, and consider the decomposition of N into disjoint orbits under this action. The orbits all have p-power order, since G has p-power order. Since N has order divisible by p, the number of orbits with one element must be divisible by p. To say that the orbit of $n \in N$ has order 1 is to say that it commutes with each element of G, i.e., to say that it lies in Z. Hence the number of elements in $N \cap Z$ is divisible by p. In particular, $N \cap Z \neq (e)$.

Undergrads: there is a competitive fellowship that you can apply for. Applications are available from Rita Torres-Gonzalez in the office behind 970 Evans. Press me for details.