Math H113 — Professor K. A. Ribet

Midterm Exam — April 8, 2003

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

These solutions were written quickly by Ken Ribet. Sorry if they're a little terse. They're perhaps best described as sketches of solutions.

1. (4 points) Show that a group of order 40 has a normal subgroup of order different from 1 and 40.

The number of 5-Sylow subgroups divides 8 and must be 1 mod 5. Therefore, this number is 1. The group has a normal subgroup of order 5.

2. (5 points) Find the number of elements of order 7 in a simple group of order $168 = 2^3 \cdot 3 \cdot 7$.

The number of 7-Sylows divides 24 and is 1 mod 7. Further, it can't be 1 because the group has no normal subgroup other than itself and the trivial group. The number in question must then be 8. Each 7-Sylow has 6 elements of order 7 and each element of order 7 lies in a unique 7-Sylow subgroup. We conclude that there are $8 \times 6 = 48$ *elements of order 7 in the group.*

3. (7 points) Let G be a finite group of p-power order. Let N be a normal subgroup of G of order p. Prove that N is contained in the center of G.

The group G *operates on* N *by conjugation; the operation is given by a homomorphism* $\alpha : G \to \operatorname{Aut} N$, *where* $\operatorname{Aut} N$ *is the group of group automorphisms of* N. *We know that* N *is cyclic of order* p, *so* $\operatorname{Aut} N$ *may be viewed as the group of non-zero integers mod* p. *This group has order* $p-1$, *a number that is prime to* p. *The image of* α *is a quotient of* G, *so it has* p*-power order. Also, it is a subgroup of* $\operatorname{Aut} N$, *so its order divides*

$p-1$. This image must therefore be trivial, since it its order is a power of p that divides $p-1$. Accordingly, α is the trivial homomorphism. This means that all elements of G commute with elements of N. Conclusion: N lies in the center of G.

4. (7 points) Consider the cycle $\sigma = (1\,2\,\cdots\,n-1\,n)$ in S_n. Show that σ has $(n-1)!$ conjugates and that the centralizer of σ has order n.

The conjugates of σ are exactly the n-cycles of S_n. Indeed, conjugating σ by τ yields the n-cycle $(\tau(1)\,\tau(2)\cdots\tau(n))$, which can be anything we want. The number of n-cycles in clearly $(n-1)!$; it's the number of ways of arranging n letters in a line, divided by n. (We divide by n because each n-cycle can be written in exactly n different ways.) Now the conjugacy class of an element g of a group G is canonically the same thing as $G/C_G(g)$; looking at orders, we conclude in this case that the centralizer of σ has order n. This centralizer contains the powers of σ, of which there are n, so it contains nothing more.

5. (7 points) Let S and T be subsets of a finite group G. Suppose that $|S|+|T|>|G|$. Show, for each $g\in G$, that the intersection of S with $gT^{-1}=\{\,gt^{-1}\,|\,t\in T\,\}$ is non-empty. Prove that G coincides with its subset $ST=\{\,st\,|\,s\in S, t\in T\,\}$.

The set gT^{-1} has the same number of elements as T; indeed, it is in bijection with T under the map $t\in T\mapsto gt^{-1}\in gT^{-1}$. Thus S and gT^{-1} cannot be disjoint, since the sum of their sizes is bigger than the size of G. This means that the two sets have an element in common, i.e., that there are $s\in S$ and $t\in T$ so that $s=gt^{-1}$. This gives $g=st$, so that g lies in ST. Since g was an arbitrary element of G, we get $G=ST$, as required.