If you get a chance, I'd love to see solutions to §4.4 #5 and/or §4.5 #16. Thanks. Hope you have a relaxing break.

Problem 5 of §4.4: Think first about D_8, which we view as usual with its generators r (order 4) and s (order 2). An automorphism α of D_8 must take r to an element of order 4. There are only two such elements, namely r and r^{-1}. Let β be the automorphism $g \mapsto sgs$, which is of order 2. Then β takes r to r^{-1}, so either α or $\beta\alpha$ takes r to r. An automorphism that fixes r (and its powers) must take s to an element of order 2 other than r^2. There are four such elements (namely sr^i with $0 \leq i \leq 3$), so D_8 has at most four automorphisms that fix r. Because each automorphism either fixes r or is β composed with an automorphism that fixes r, there are at most 8 automorphisms of D_8. Now D_8 has a non-trivial center, and in fact its center must have order 2 because $G/Z(G)$ cannot be cyclic unless G is an abelian group. The center of D_8 is the group $\{1, r^2\}$. It follows that the group of inner automorphisms of D_8 is a group of order 4; it is isomorphic to $D_8/Z(D_8)$, which is a non-cyclic group of order 4 (Klein four group). The inner automorphisms of D_8 include the map β introduced above and the inner automorphism $\gamma :=$ "conjugation by r," which sends r to r and s to sr^2. We have also the identity map and the automorphism $\gamma\beta$.

To obtain more automorphisms of D_8, we should embed D_8 into D_{16} as suggested by the wording of the problem. If we regard D_{16} as generated by an element x of order 8 together with a symmetry s of order 2 (with the usual relation $sxs = x^{-1}$), then the subgroup of D_{16} generated by s and $r := x^2$ is a copy of D_8. The D_8 inside D_{16} is certainly normal, since it is a subgroup of index 2, so conjugations by elements of D_{16} yield automorphisms of D_8 that are no longer necessarily inner. In particular, conjugation by x yields the (non-inner) automorphism α that sends r to r but sends s to sr^{-1}. This new automorphism α is of order 4; the automorphisms α and β (which has order 2) generate a group of automorphisms of D_8 that has order 8. You can check easily that $(\alpha\beta)^2$ is the identity map on D_8, so that the group generated by α and β is a dihedral group (and thus isomorphic to D_8). Note that the identity $(\alpha\beta)^2 = 1$ may be rewritten $\beta\alpha\beta = \alpha^{-1}$ because β has order 2. This latter re-writing makes it clear that we are dealing with a dihedral group.

§4.5, problem 16: The number of Sylow subgroups of order r is congruent to 1 mod r and is a divisor of pq. It can't be p or q because these are less than r. So it's either 1 or pq. In the former case, we're done, so let's assume that there are pq different subgroups of order r. The number of elements of order r is then $pq(r-1) = pqr - pq$. Thus there are only pq different elements of order other than r. Consider the q-Sylow subgroups. The number of them divides pr, so there are at least p different Sylow subgroups of order q if there is not a unique q-Sylow. In other words, if there is not a normal q-Sylow subgroup, then there are at least $p(q-1)$ different elements of order q. If there is no normal r-Sylow or q-Sylow, then the number of group elements not of order r or q is at most $pq - p(q-1) = p$. In this situation, there can be only one p-Sylow subgroup: there cannot be more than p elements of order 1 or p.