Math 110

PROFESSOR KENNETH A. RIBET

First Midterm Examination

February 20, 2014

9:40–11:00 AM, 105 Stanley Hall

Please write your NAME clearly: Ken Ribet.

These are skeletal solutions, written quickly after the exam ended.

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are vector spaces over $\mathbf{F}$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

At the conclusion of the exam, hand your paper in to your GSI.

Problem	Your score	Possible points
1		5 points
2		6 points
3		6 points
4		6 points
5		7 points
Total:		30 points

1. Consider the basis $(1, x, x^2, x^3)$ of the $\mathbf{R}$-vector space $V = \mathcal{P}_3(\mathbf{R})$. Let $(\varphi_1, \ldots, \varphi_4)$ be the basis of V' that is dual to $(1, x, x^2, x^3)$. Let $\varphi : V \to \mathbf{R}$ be the linear functional

$$f(x) \mapsto f(6) + \int_0^1 f(x)\, dx.$$

Find numbers a, b, c, d for which $\varphi = a\varphi_1 + b\varphi_2 + c\varphi_3 + d\varphi_4$.

Suppose that we apply φ to one of the basis vectors; let's apply it to x. Then $\varphi(x) = a\varphi_1(x) + b\varphi_2(x) + c\varphi_3(x) + d\varphi_4(x) = a \cdot 0 + b \cdot 1 + c \cdot 0 + d \cdot 0$, with the last equality coming from the definition of a dual basis. Hence $b = \varphi(x) = 6 + \dfrac{1}{2}$. The other constants can be recovered by evaluating φ on the other basis vectors.

2. Label each of the following assertions as TRUE or FALSE. Along with your answer, provide an informal proof, counterexample or other explanation.

a. If $T : V \to W$ is a linear map and $v_1, v_2, \ldots, v_r$ are vectors of V such that $(Tv_1, \ldots, Tv_r)$ is linearly independent, then $(v_1, \ldots, v_r)$ is linearly independent.

This is true. The linear independence of $(v_1, \ldots, v_r)$ means that the equation $0 = a_1v_1 + \cdots + a_rv_r$ implies that all the a_i are 0. The equation implies that $0 = a_1Tv_1 + \cdots + a_rTv_r$ by the definition of "linear map." The latter equation implies that the a_i are 0 because the Tv_i are linearly independent.

b. If $T : V \to W$ is a linear map and $v_1, v_2, \ldots, v_r$ are vectors of V such that $(Tv_1, \ldots, Tv_r)$ spans W, then $(v_1, \ldots, v_r)$ spans V.

This is blatently false. For example, W could be the 0-vector space, so that T would take all elements of V to 0. Any old random list of vectors in V will be such that their images span $W = \{0\}$, but the vectors in V don't have to span V.

3. Label each of the following assertions as TRUE or FALSE. Along with your answer, provide an informal proof, counterexample or other explanation.

a. If X is a 5-dimensional subspace of a 8-dimensional vector space V, there is a 2-dimensional subspace Y of V such that $X \cap Y = \{0\}$.

This is true. The simplest thing to do is to pick a basis $(v_1, \ldots, v_5)$ of X and extend it to a basis $(v_1, \cdots, v_5, v_6, v_7, v_8)$ of V. We can take Y to be the span of (v_6, v_7). This is like choosing a complementary subspace to X (as we did concretely in a lot of homework problems), except that we use only two of the "extra" vectors instead of using all three.

b. If $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ are linearly independent lists of vectors in V, then $(v_1 + w_1, \ldots, v_m + w_m)$ is linearly independent.

This is false, and kind of silly. (It's also an exercise in the book.) We could take the w_i to be the negatives of the v_i; the sums in the list would all be 0.

4. Suppose that $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(-1) = 0$ for all j. Prove that $(p_0, p_1, \ldots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

If the given $m + 1$ polynomials were linearly independent, they would form a basis of the $(m + 1)$-dimensional vector space $\mathcal{P}_m(\mathbf{F})$. Then all polynomials of degree $\leq m$ would be linear combinations of the p_j. This would give that all such polynomials $p(x)$ of degree $\leq m$ satisfy $p(-1) = 0$. But clearly there are polynomials that do not have this property, for example the constant polynomial 1.

5. Let U be a subspace of V and let $T : U \to W$ be a non-zero linear map. Suppose that the function

$$S(v) := \begin{cases} T(v) & \text{if } v \in U \\ 0 & \text{if } v \notin U \end{cases}$$

is a linear map $V \to W$. Show that $U = V$.

Let's assume that U is smaller than V and that S is linear. We'll show that T is identically 0. If we can accomplish this, then we'll have proved the desired assertion; this would be a proof by contradiction, if you wish, since T is assumed going in to be a non-zero linear map.

The assumption that U is smaller than V means that there is a vector in V that's not in U. Fix such a vector; call it v. For each $u \in U$, we have

$$T(u) = S(u) = S((u - v) + v) = 0 + 0 = 0,$$

as desired. The point here is that $u - v$ and v are vectors in V that are not in U.