Math 110 — Professor K. A. Ribet

Midterm Exam — February 22, 2005

This exam was an 80-minute exam. It began at 12:40PM. There were 4 problems, for which the point counts were 12, 8, 8 and 8. The maximum possible score was 36.

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded. At the conclusion of the exam, please hand in your paper to your GSI.

According to the GSIs, there were 136 people at the exam. The median score is somewhere between 16.5 and 20. The average is something like 19.2. Here is a quick summary of how many papers were in different ranges of scores:

32.5–36: 5
28.5–32: 8
24.5–28: 21
20.5–24: 23
16.5–20: 35
12.5–16: 27
8.5–12: 13
4.5–8: 3
0–4: 1

If you want to know how your score might convert into a letter grade, you can recall the statement that I made about final grades in Fall, 2002: "I awarded 61 grades as follows: 15 As, 20Bs, 13Cs, 12Ds and 1F. Those students who took the course on a P/NP basis had their letter grades converted into either P or NP. I awarded Fs to those students who were signed up for the course but did not take the final exam. I believe that all of them left the course early on in the semester and had forgotten to drop the class."

1. *Label the following statements as TRUE or FALSE, giving a short explanation (e.g., a proof or counterexample). There are six parts to this problem, which continues on page 3.*

a. *If A and B are $n \times n$ matrices over F, then $AB = 0$ if and only if $BA = 0$.*

This looks severely FALSE to me. If I'm not mistaken, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is non-zero and the product of these matrices taken in the other order is 0.

b. *If A and B are $n \times n$ matrices over the field of real numbers, then $AB = 2I_n$ if and only if $BA = 2I_n$.*

TRUE because we're saying that A is the left inverse of $\frac{1}{2}B$. See §2.4, problem 9 and my solution to it (on the web page). This works only because there is a $\frac{1}{2}$ in our field. If 2 were equal to 0 in our field, then we'd be back in part (b) of this question and the answer would be FALSE. If F were given as a field and there were no hypothesis about the invertibility of the number 2, then the answer would again be FALSE for the same reason.

c. *If $x, y \in V$ and $a, b \in F$, then $ax + by = 0$ if and only if x is a scalar multiple of y or y is a scalar multiple of x.*

This is complete nonesense; I was trying to imitate the spirit of our authors' questions. For example, a and b could both be 0. This one is FALSE.

d. *For $A \in M_{m \times n}(F)$, $[L_A]_\beta^\gamma = A$ if β and γ are the standard bases of F^n and F^m.*

TRUE, basically by definition of L_A and by the definition of how you compute the matrix of a linear transformation.

e. *Every vector space over F is isomorphic to F^n for some $n \geq 1$.*

This is FALSE because the vector space might be $\{0\}$, for instance. The vector space could also be infinite-dimensional.

f. *If v is a non-zero vector in a finite-dimensional vector space V, then there is a linear form $f \in V^*$ such that $f(v) = 1$.*

TRUE as explained in the proof of the lemma on page 122 of the book and in my lecture last week.

2. *Suppose that $T : V \to V$ is a linear transformation on an F-vector space V. Prove that $T^2 = 0$ if and only if the range of T is contained in the null space of T.*

This question was put on the exam in order to give you a hint for the question that follows. To say that the range is contained in the null space is to say that T vanishes on all vectors of the form Tv with $v \in V$, i.e., that $T(Tv) = 0$ for all v. Thus the range is contained in the null space if and only if $T^2(v) = 0$ for all $v \in V$, which is the statement that $T^2 = 0$.

Assume that V is finite-dimensional and that $T^2 = 0$. Prove that $\operatorname{nullity}(T) \geq (\dim V)/2$.

Under the assumption $T^2 = 0$, the nullity of T is at least as big as the rank of T, since the range of T is contained in the null space of T. Since the rank plus the nullity sum up to $\dim V$, twice the nullity is at least as big as $\dim V$.

3. *Let $V = P(\mathbf{R})$ be the vector space of polynomials with real coefficients. For each $i \geq 0$, let $f_i \in \mathcal{L}(V, \mathbf{R})$ be the linear transformation that maps a polynomial $p(x)$ to the value $p^{(i)}(0)$ of its ith derivative at 0. (The 0th derivative of p is p itself.) Show that the linear functionals $f_0, f_1, f_2, \ldots$ are linearly independent. [Evaluate linear combinations of the f_i on powers of x.]*

It's important here to understand what is meant by the linear independence of an infinite set of elements of a vector space (which in this case we take to be $\mathcal{L}(V, \mathbf{R})$): it means that there is no non-trivial linear relation among a finite set of the elements. To say that the infinite set $\{f_0, f_1, f_2, \ldots\}$ is linearly independent is to say that a linear combination $a_1 f_1 + \cdots a_n f_n$ (with $n \geq 1$) can vanish only when all of the coefficients a_i are 0. Assume that we have $0 = a_1 f_1 + \cdots a_n f_n$. For each $i = 1, 2, \ldots, n$, evaluate $a_1 f_1 + \cdots a_n f_n$ at x^i, as suggested by the hint. The point is that $f_j(x^i)$ is $i!$ if $j = i$ and 0 otherwise. We thus get $a_i i!$ when we evaluate $a_1 f_1 + \cdots a_n f_n$ at x^i. Since this value is 0 because $0 = a_1 f_1 + \cdots a_n f_n$, we obtain $a_i = 0$ on dividing by $i!$.

Show that the linear transformation $g : p(x) \mapsto p(1)$ is not in the span of the f_i.

This is similar to the first part of the problem. Suppose that g is in the span; then it's equal to $a_1 f_1 + \cdots a_n f_n$ for some choice of n and some coefficients a_i. If we exaluate $a_1 f_1 + \cdots a_n f_n$ on x^{n+1}, we get 0. If we evaluate g on x^{n+1}, we get 1. Since 1 and 0 are not equal, we have a contradiction.

4. *Let $T : V \to W$ be a linear transformation between F-vector spaces. Suppose that $x_1, \ldots, x_r$ are linearly independent elements of $N(T)$ and that $v_1 \ldots, v_s$ are vectors in V such that $T(v_1), \ldots, T(v_s)$ are linearly independent. Show that the $r+s$ vectors $x_1, \ldots, x_r; v_1, \ldots, v_s$ are linearly independent.*

This question was inspired by my solution to problem 16 in §2.2. Suppose that we have a linear dependence relation among the vectors $x_1, \ldots, x_r$, $v_1, \ldots, v_s$: $0 = \sum a_i x_i + \sum b_j v_j$. Take T of both sides; we get $0 = \sum b_j T(v_j)$ because $T(x_i) = 0$ for all i. Because the v_j are linearly independent, all b_j are 0. Thus we have $0 = \sum a_i x_i$. Since the x_i are linearly independent, the a_i are 0 as well.