Math 110 Professor K. A. Ribet

Final Exam May 18, 2005

This exam was a 180-minute exam. It began at 5:00PM. There were 7 problems, for which the point counts were 8, 9, 8, 7, 8, 7, and 7. The maximum possible score was 54.

*Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded. At the conclusion of the exam, please hand in your paper to your GSI.*

**1.** *Let $T$ be a linear operator on a vector space $V$. Suppose that $v_1, \dots, v_k$ are vectors in $V$ such that $T(v_i) = \lambda_i v_i$ for each $i$, where the numbers $\lambda_1, \dots, \lambda_k$ are distinct elements of $F$. If $W$ is a $T$-invariant subspace of $V$ that contains $v_1 + \cdots + v_k$, show that $W$ contains $v_i$ for each $i = 1, \dots, k$.*

See my solutions for homework set #11.

**2.** *Assume that $T : V \to W$ is a linear transformation between finite-dimensional vector spaces over $F$. Show that $T$ is 1-1 if and only if there is a linear transformation $U : W \to V$ such that $UT$ is the identity map on $V$.*

One direction is obvious; if $UT = 1_V$ and $T(v) = 0$, then $v = U(T(v)) = 0$, so that $T$ must be injective. The harder direction is to construct $U$ when $T$ is given as 1-1. Choose a basis $v_1, \dots, v_n$ of $V$ and let $w_i = T(v_i)$ for each $i$. Because $T$ is injective, the $w_i$ are linearly independent. Complete $w_1, \dots, w_n$ to a basis $w_1, \dots, w_n; w_{n+1}, \dots, w_m$ of $W$. We can define $U : W \to V$ by declaring the images $U(w_i)$ of the basis vectors $w_i$; if we want $w_i$ to go to $x_i \in V$, then we define $U(\sum a_i w_i) = \sum a_i x_i$. We take $x_i = v_i$ for $i = 1, \dots, n$ and take (for instance) $x_i = 0$ for $i > n$. It is clear that $(UT)(v_i) = v_i$ for each basis vector $v_i$ of $V$. It follows from this that $UT$ is the identity map on $V$.

**3.** *Let $T$ be a self-adjoint linear operator on a finite-dimensional inner product space $V$ (over $\mathbf{R}$ or $\mathbf{C}$). Show that every eigenvalue of $T$ is a positive real number if and only if $\langle T(x), x \rangle$ is a positive real number for all non-zero $x \in V$.*

Because $T$ is self-adjoint, there is an orthonormal basis $\beta$ of $V$ in which $T$ is diagonal. Let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of the diagonal matrix $[T]_\beta$. The $\lambda_i$ are real numbers even if $V$ is a complex vector space. The issue is whether or not these real numbers are all positive. If $x$ has coordinates $a_1, \ldots, a_n$ in the basis $\beta$, then $\langle T(x), x\rangle = \sum_i |a_i|^2 \lambda_i$. In particular, we can take $x$ to be the $i$th element of $\beta$, so that its $i$th coordinate is 1 and its other coordinates are 0. Then $\langle T(x), x\rangle = \lambda_i$. Thus if $\langle T(x), x\rangle$ is always positive, $\lambda_i$ is positive for each $i$. Conversely, if the $\lambda_i$ are positive, the sum $\sum_i |a_i|^2 \lambda_i$ is non-negative for all $n$-tuples $(a_1, \ldots, a_n)$ and is positive whenever $(a_1, \ldots, a_n)$ is non-zero, i.e., whenever the vector $x$ corresponding to $(a_1, \ldots, a_n)$ is a non-zero element of $V$.

**4.** *Let $V$ be an inner product space over $F$ and let $X$ and $Y$ be subspaces of $V$ such that $\langle x, y\rangle = 0$ for all $x \in X$, $y \in Y$. Suppose further that $V = X + Y$. Prove that $Y$ coincides with $X^\perp = \{\, v \in V \mid \langle x, v\rangle = 0 \text{ for all } x \in X \,\}$.*

Under the assumptions of the problem, everything in $Y$ is perpendicular to everything in $X$, so we have $Y \subseteq X^\perp$. If $v$ is perpendicular to all vectors in $X$, we must show that $v$ lies in $Y$. Because $V = X + Y$, we may write $v = x + y$ with $x \in X$, $y \in Y$. We have $0 = \langle v, x\rangle = \langle x + y, x\rangle = \langle x, x\rangle + \langle y, x\rangle = \langle x, x\rangle + 0 = \langle x, x\rangle$. Because $\langle x, x\rangle = 0$, $x = 0$ by the axioms of an inner product. Hence $v = y$ does indeed lie in $Y$. (This problem was inspired by a comment of Chu-Wee Lim, who pointed out to me that the definition on page 398 of the book, and the comments following the definition, are extremely bizarre.)

**5.** *Let $V$ be the space of polynomials in $t$ with real coefficients. Use the Gram–Schmidt process to find non-zero polynomials $p_0(t), p_1(t), p_2(t), p_3(t)$ in $V$ such that $\int_{-1}^{1} p_i(t)p_j(t)\,dt = 0$ for $0 \le i < j \le 3$. (It may help to note that $\int_{-1}^{1} t^i\,dt = 0$ when $i$ is an odd positive integer.)*

There is an implicit inner product here: $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$. The vectors 1, $t$, $t^2$, and $t^3$ are linearly independent elements of $V$ and we can apply G–S to this sequence of vectors to generate an orthogonal set of vectors; this is what the problem asks for. The computations, which I won't reproduce, are easy because of the remark about the integrals of odd powers of $t$. The answer that I got is that the $p_i$ are in order: 1, $t$, $t^2 - \frac{1}{3}$, $t^3 - \frac{3}{5}t$. I did these computations in class; the $p_i$ are called Legendre polynomials.

**6.** *Let $A$ be an $n \times n$ matrix over $F$ and let $A^t$ be the transpose of $A$. Using the equality "row rank = column rank," show for each $\lambda \in F$ that the vector spaces $\{\, x \in F^n \mid Ax = \lambda x \,\}$ and $\{\, x \in F^n \mid A^t x = \lambda x \,\}$ have the same dimension.*

Combining the "row rank = column rank" theorem with the formula relating rank and nullity, we see that the linear transformations $L_A$ and $L_{A^t}$ have equal nullities. These nullities are the dimensions of the null spaces of the two transformations; meanwhile, the null spaces are exactly the two vector spaces of the problem in the case $\lambda = 0$. To treat the case of arbitrary $\lambda$, we have only to replace $A$ by $A - \lambda I_n$.

**7.** *Let $A$ be an element of the vector space $M_{n\times n}(F)$, which has dimension $n^2$ over $F$. Show that the span of the set of matrices $\{\, I_n, A, A^2, A^3, \dots \}$ has dimension $\leq n$ over $F$.*

Direct application of Cayley–Hamilton: the set $\{\, I_n, A, A^2, A^3, \dots, A^{n-1} \,\}$ has the same span as the full set of all powers of $A$.