The lecture will be full of matrices and formulas. Here is a sketch of what I intend to talk about. Especially if you read this document before the lecture, you can take no notes or fewer notes than usual.

We work with vector spaces over a field $F$. Most are finite-dimensional. Take vector spaces $V$ and $W$ of finite dimension and suppose that we fix ordered bases $\beta$ and $\gamma$ of $V$ and $W$, respectively. We have a beautiful association $L(V, W) \rightarrow M_{m \times n}(F)$ given by $T \mapsto [T]_{\beta}^{\gamma}$. The source and target $L(V, W)$ and $M_{m \times n}(F)$ are $F$-vector spaces in a natural way: we know how to add matrices and how to multiply matrices by scalars. Similarly, we know how to add linear transformations and multiply them by scalars—we discussed this on February 4 (Thursday). The first point is that $T \mapsto [T]_{\beta}^{\gamma}$ is a linear map (= linear tranformation) between vector spaces. This means that the sum of two $T$s goes to the sum of their matrices and that $[aT]_{\beta}^{\gamma} = a[T]_{\beta}^{\gamma}$ for $a \in F$ and $T \in L(V, W)$. These identities come from the definition of $[T]_{\beta}^{\gamma}$ that we gave on Thursday.

The claim is that $T \mapsto [T]_{\beta}^{\gamma}$ is an isomorphism of $F$-vector spaces. A linear map is said to be an isomorphism when it’s invertible. This means that it’s 1-1 and onto; we discussed invertible maps in class on Thursday. To see that the map is 1-1, we have to check that its null space is 0, i.e., that $[T]_{\beta}^{\gamma} = 0$ implies that $T = 0$. If $[T]_{\beta}^{\gamma} = 0$, then the construction of $[T]_{\beta}^{\gamma}$ shows that $T(v_j) = 0$ for all basis vectors $v_j \in \beta$. Since $T$ is linear, $T = 0$. To see that the map is surjective (onto), we suppose that we are given a matrix $A = (a_{ij})$ in $M_{m \times n}(F)$. For each $j$, $1 \leq j \leq n$, let $y_j$ be the element of $W$ dictated to us by the $j$th column of $A$, namely $\sum_{i=1}^{m} a_{ij}w_i$, and then let $T$ be the unique linear map $V \rightarrow W$ that sends $v_j$ to $y_j$. (The existence of this map was discussed last week, especially on Thursday.) It is clear from the definition of $[T]_{\beta}^{\gamma}$ that this matrix coincides with the given matrix $A$.

One consequence is that the dimension of $L(V, W)$ is $mn$, since $mn$ is clearly the dimension of the space of $m \times n$ matrices over $F$. Remember that isomorphic vector spaces have equal dimensions.

By the way, here’s a digression. If $V$ has dimension $n$, then we get an isomorphism $F^n \sim V$ by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i v_i \in V$. If $V$ and $V'$ both have dimension $n$, then they are each isomorphic to $F^n$, so they are isomorphic to each other. Two finite-dimensional vector spaces are isomorphic if and only if they have equal dimensions. If $V$ and $W$ are vector spaces, we could write $V \sim W$ as an abbreviation for the statement “there exists some isomorphism from $V$ to $W$.” The statement means that $V$ and $W$ are isomorphic. The relation of being isomorphic is an equivalence relation. If $T : V \rightarrow W$ is a linear map between finite dimensional vector spaces of the same dimension and if $U : W \rightarrow V$ is a linear map such that $UT = 1_V$, then $T$ and $U$ are invertible and they are inverses of each other. End of digression.
The next theme is that of matrices of compositions. If we have $T : V \rightarrow W$ and $U : W \rightarrow Z$, we get $UT$ (also written $U \circ T$), which means “$T$ followed by $U$.” It’s a map $V \rightarrow Z$ and we know by now that it’s a linear map. Formally, we could describe composition as a mapping $\mathcal{L}(W, Z) \times \mathcal{L}(V, W) \rightarrow \mathcal{L}(V, Z)$.

Assume that we have ordered bases $\alpha = \{v_1, \ldots, v_n\}$, $\beta = \{w_1, \ldots, w_m\}$, $\gamma = \{z_1, \ldots, z_d\}$ of $V$, $W$ and $Z$, respectively. The big claim is as follows:

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$ 

On the right-hand side we have the product of a $d \times m$ matrix and an $m \times n$ matrix. On the left we have a $d \times n$ matrix. The dimensions are compatible with our identity being both meaningful and true. It remains to compute things and check that everything works.

For the matrix $T$, we use familiar notation. For each $j = 1, \ldots, n$, we write $Tv_j = \sum_{i=1}^m a_{ij}w_i$ and then say that $[T]_\alpha^\beta = A$ with $A = (a_{ij})$. Note that I write $Tv_j$ instead of $T(v_j)$; omitting parentheses looks good here. Define $[U]_\beta^\gamma = B = (b_{ki})$ by the analogous formula; namely, write $Uw_i = \sum_{k=1}^d b_{ki}z_k$ for each $i$. Finally, introduce $[UT]_\alpha^\gamma = C = (c_{kj})$ by writing $UT(v_j) = \sum_k c_{kj}z_k$ for each $j$. Because of the linear independence of the $z_k$, the $c_{kj}$ are the unique elements of $F$ that satisfy these identities. What we need to check is that $C = BA$, which means that $c_{kj} = \sum_i b_{ki}a_{ij}$ for each $k$ and $j$. It is enough to show that

$$UT(v_j) = \sum_k \sum_i b_{ki}a_{ij} z_k$$

for each $j$.

Now

$$UT(v_j) = U(\sum_i a_{ij}w_i) = \sum_i a_{ij} U(w_i) = \sum_i a_{ij} \sum_k b_{ki} z_k.$$ 

After rearranging the right-hand sum a tiny bit, we get the desired formula.

The formula that we have just proved actually specializes to a formula that we proved in the waning minutes of class on Thursday. This is interesting: Suppose that $x$ is a vector in an $n$-dimensional vector space $V$. Consider the linear map $T : F \rightarrow V$ taking $a \in F$ to $ax \in V$. The vector space $F = F^1$ has the standard basis $\{1\}$. If we use this 1-element basis and an $n$-element basis $\beta = \{v_1, \ldots, v_n\}$ of $V$, we get a matrix $[T]_\beta^\beta$, which is in fact an $n \times 1$ matrix—it’s a column of length $n$. We see immediately from the definitions that $[T]_\beta^\beta = [x]_\beta$, where the right-hand side of the equation is the column that expresses $x$ in terms of $\beta$. Now suppose that $U : V \rightarrow W$ is a linear transformation and that we have an ordered basis $\gamma$ of $W$. Then we may write

$$[UT]_{\{1\}}^\gamma = [U]_\beta^\gamma [T]_{\{1\}}^\beta.$$
by the general formula that we’ve proved. Now $UT$ is the map $F \to W$ taking $a$ to $U(x)$, so it’s the analogue of $T$ with $x \in V$ replaced by $Ux \in W$. Thus the displayed formula becomes $[Ux]_\gamma = [U]_\beta^\gamma [x]_\beta$, which is what we proved in class at the end of Thursday’s lecture. (The map from $V$ to $W$ was called $T$ instead of $U$ in that lecture.)

We have seen that linear transformations give matrices and that every matrix of the right size comes from an element of $L(V,W)$ when we have in the picture two spaces $V$ and $W$ with fixed bases $\beta$ and $\gamma$. Another thing we can do is to take a matrix cold and not have any vector spaces around. Take $A \in M_{m\times n}(F)$ and notice that left-multiplication by $A$ gives a map $F^n \to F^m$, $x \mapsto Ax$. Here, we think of $F^n$ and $F^m$ as spaces of column vectors. This map $F^n \to F^m$ is called $L_A$ by our authors; the “$L$” could stand for “left” (as in multiplication on the left) or “linear.” I’m pretty sure that they had “linear” in mind. It’s easy to check that $L_A$ is a linear map; this follows mainly from the distributive law for matrix multiplication, since we have to recognize that $A(x + x') = Ax + Ax'$. If $\beta$ and $\gamma$ are the standard bases on $F^n$ and $F^m$, then $[L_A]_\beta^\gamma = A$. (You all probably remember that the standard basis vectors have one 1 and otherwise consist of 0s.)

Now we get to a super-important topic: change of basis. This really is just an application of compositions. Suppose that we have a map $T : V \to W$ between finite-dimensional $F$-vector spaces and that $V$ and $W$ have bases $\beta$ and $\gamma$. The wrinkle now is that we assume that $V$ has a second, alternative basis $\beta'$. We would like to compare $A := [T]_\beta^\gamma$ and $[T]_{\beta'}^\gamma$. Imagine that we understand how to write $\beta'$ in terms of $\beta$. In other words, imagine that $\beta = \{v_1, \ldots, v_n\}$ and $\beta' = \{v'_1, \ldots, v'_n\}$ and that we know how to express each $v'_j$ as a linear combination of the $v_i$: $v'_j = \sum_i q_{ij}v_i$ for each $j$. The key insight is that the matrix $Q = (q_{ij})$ is nothing but $[1_V]_\beta^\beta'$; this really follows pretty directly from definitions in our setup. Using the formula for the matrix of a composite, we get

$$[T]_{\beta'}^\gamma = [T \circ 1_V]_{\beta'}^\gamma = [T]_\beta^\gamma [1_V]_{\beta'}^\beta = AQ.$$  

When we change from $\beta$ to $\beta'$, we multiply $A = [T]_\beta^\gamma$ on the right by the change-of-basis matrix $Q$.

Assume finally that $W$ has a second basis $\gamma'$ and let $R = [1_W]_{\gamma'}^\gamma$ be the analogue of $Q$; it’s the matrix that expresses the vectors in $\gamma'$ in terms of those of $\gamma$. We see similarly that

$$[T]_{\beta'}^\gamma' = [1_W]_{\gamma'}^\gamma [T]_{\beta}^\gamma [1_V]_{\beta'}^\beta.$$  

In the product on the right-side of the equation, we have already understood the second and third factors and need to identify the first factor $[1_W]_{\gamma'}^\gamma$. For various reasons that I’m not motivated to type into this file, $[1_W]_{\gamma'}$ is nothing but the inverse of the matrix $R = [1_W]_{\gamma'}$. Thus the matrix of $T$ with respect to the new bases $\beta'$ and $\gamma'$ is simply $R^{-1}AQ$ where $R$ and $Q$ are the transition matrices between the two $\beta$s and the two $\gamma$s (as described above) and $A$ is the matrix in the original bases.