Math 110
Notes for the lecture on February 10, 2005

The first part of the lecture will correspond to the end of the notes that were posted for February 8. I will discuss change of basis. For details on this, see the notes for February 8. The situation there is that we have a linear $T : V \to W$, where V and W are finite-dimensional. We assume that V has bases β and β' and that, analogously, W has bases γ and γ'. Imagine that we can write β' in terms of β and γ' in terms of γ. Then there are three obvious matrices lying around:

$$A = [T]_\beta^\gamma, \qquad Q = [1_V]_{\beta'}^{\beta}, \qquad R = [1_W]_{\gamma'}^{\gamma}.$$

Then the formula to remember is that

$$[T]_{\beta'}^{\gamma'} = R^{-1}AQ.$$

As I said, details are in the last set of notes.

A very important example is that where $W = V$, $\gamma = \beta$ and $\gamma' = \beta'$. The book writes $[T]_\beta$ for $[T]_\beta^\beta$ and employs analogous notation for $[T]_{\beta'}^{\beta'}$. We then get the formula

$$[T]_{\beta'} = R^{-1}[T]_\beta R,$$

which is very important. The formula states that $[T]_{\beta'}$ is the *conjugate* of $[T]_\beta$ by R; the word "conjugate" is undoubtedly familiar to you if you've taken Math 113.

For an example, take F to be the field of complex numbers and let $V = W = F^2$. Let β be the standard basis of V. Let a and b be complex numbers; let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Let $T = L_A$. Then of course $[T]_\beta = A$. Let β' be the alternative basis $\{(1, -i), (1, +i)\}$. Then we should be able to check in class that $[T]_{\beta'}$ is the 2×2 diagonal matrix whose diagonal entries are $a + bi$ and $a - bi$. The matrix R here is $\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$.

Just to answer someone's question: we are not going to discuss §2.7. We will, however, discuss §2.6. This section concerns the all-important topic of **dual spaces**. If V is an F-vector space, its dual V^* is the space $\mathcal{L}(V, F)$ of linear transformations $V \to F$. Such linear transformations are called *linear functionals*. If V is finite dimensional and has (ordered) basis $\beta = \{v_1, \ldots, v_n\}$, then there are n different elements of V^* staring us in the face. These are the coordinate functions $f_1, \ldots, f_n$ that are defined by the basis β. Namely, for $v \in V$, we may write uniquely $v = \sum_{i=1}^n a_i v_i$ with $a_i \in F$. The functional f_i maps v to the coefficient a_i. It is easy to see that f_i is linear. It satisfies the key formula $f_i(v_j) = \delta_{ij}$, where δ_{ij} is the *Kronecker delta* function, which by definition is 1 when $i = j$ and 0 when i and j are distinct.

For $f \in V^*$, we see that f is determined by the numbers $f(v_i)$ for $i = 1, \ldots, n$. This follows from a general theorem, which states that a linear map $T : V \to W$ is determined by the vectors $T(v_i)$ in W. Explicitly here: if $v = \sum a_i v_i$ as before, then

$$f(v) = \sum_{i=1}^{n} a_i f(v_i).$$

Once we know the $f(v_i)$ we have a recipe for finding f(anything): write each vector of V in terms of the basis vectors and use the formula that's displayed just above.

As the book points out, V^* has dimension n when V has dimension n. Indeed, $\mathcal{L}(V, W)$ is known by us to have dimension nm when V has dimension n and W has dimension m. Here, W is the 1-dimensional space $F = F^1$. A more precise result is that the f_i form a basis of V. This basis depends on $\beta = \{v_1, \ldots, v_n\}$ (of course) and is said to be the basis of V^* that is *dual* to the basis β. The basis dual to β is denoted β^*.

Let's prove this result. First, we should check that the f_i are linearly dependent. Suppose that we have $\sum_i c_i f_i = 0$. Then, by definition, we have that $\sum_i c_i f_i(v) = 0$ for all $v \in V$. If we put $v = v_j$, where j is a number between 1 and n, then the sum collapses to c_j because of the Kronecker delta business. Thus we have $c_j = 0$ for each j; thus the vanishing linear combination $\sum_i c_i$ was the trivial linear combination (with all coefficients 0), and we have established the required linear independence. Now let's show that each $f \in V^*$ is a linear combination of the f_i. Let f be given, and set $c_i = f(v_i)$ for each i. Then the claim is that $f = \sum_i c_i f_i$. To see this, we need to show that the difference between the two sides of the equation, which is an element of V^*, vanishes on every $v \in V$. (This forces the difference to be the 0 element of V^*.) Said differently, we want the null space of the difference to be all of V. However, the Kronecker delta business shows that the null space of the difference contains each of the basis vectors v_j. Since the null space is a subspace of V, and since the basis vectors span V, the null space of the difference is indeed the entire vector space.

The next topic concerns duals of linear maps $T : V \to W$, where V and W are finite-dimensional. There's a natural map $W^* \to V^*$, which is composition with T: $g \in W^* \mapsto gT = g \circ T \in V^*$. This map is called T^* by most authors and T^t be our authors. We'll call it T^t. The lower-case t means "transpose." The reason for this terminology becomes clear if V is given with a basis β and W with a basis γ. Then we have two matrices at hand: $[T]_\beta^\gamma$ and $[T^t]_{\gamma'}^{\beta'}$. Note that the first is an $m \times n$ matrix, while the second is an $n \times m$ matrix. The fundamental result is that the two matrices are *transposes* of each other. This result is proved by direct computation; cf. p. 121 of our text. I'll do the computation at the board; keep those pencils sharp.