Math 110 Professor K. A. Ribet

Midterm Exam September 25, 2008

This was an 80-minute exam, 3:40–5PM. There were 30 points on the test, with two questions being worth 7 points and two being worth 8 points. The explanations that follow are intended to communicate the main points of each problem but might be a little skeletal. (They're more like extended hints than complete solutions.)

1. *Let $\mathcal{P}(\mathbf{R})$ be the real vector space consisting of all polynomials with coefficients in $\mathbf{R}$. Let $D : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ be the linear map $f(x) \mapsto f'(x)$ that takes a polynomial to its derivative.*

(**a.**) *Describe the null space and the range of D.*

The null space is the space of polynomials with derivative $= 0$. Those are the constant polynomials. The range is all of $\mathcal{P}(\mathbf{R})$ since every polynomial is the derivative of some polynomial. (We know how to integrate.)

(**b.**) *Find a subspace U of $\mathcal{P}(\mathbf{R})$ such that $\mathcal{P}(\mathbf{R}) = U \oplus \operatorname{null} D$.*

We can take U to be the space of polynomials whose constant terms are 0. The intersection of U and $\operatorname{null} D$ is $\{0\}$ because the only constant in U is the polynomial 0. The sum of U and the null space is all of $\mathcal{P}(\mathbf{R})$ because each polynomial is the sum of its constant term and a polynomial whose constant term is 0.

2. *Let $T : V \to W$ be a linear map between $\mathbf{F}$-vector spaces.*

(**a.**) *Suppose that $\operatorname{null} T = \{0\}$ and that $(v_1, \dots, v_n)$ is a linearly independent list in V. Show that $(Tv_1, \dots, Tv_n)$ is linearly independent in W.*

I think that this is a good problem because it requires knowledge of the definitions and some proof-writing skills. To show that $(Tv_1, \dots, Tv_n)$ is linearly independent in W, we start with the equation $0 = a_1Tv_1 + a_2Tv_2 + \cdots + a_nTv_n$ and seek to show that the coefficients a_i are all 0. Because T is linear, we can rewrite the equation as

$$0 = T(a_1v_a + \cdots + a_nv_n).$$

Since $\operatorname{null} T = \{0\}$, the vector $a_1v_a + \cdots + a_nv_n$ is 0. Since $(v_1, \dots, v_n)$ is a linearly independent list in V, all a_i are 0.

(**b.**) *Assume that $(Tv_1, \dots, Tv_n)$ is linearly independent in W for all linearly independent lists $(v_1, \dots, v_n)$ in V. Show that $\operatorname{null} T = \{0\}$.*

The assumption is that T sends independent lists to independent lists—there is no restriction on the size of the lists. In particular, it sends independent lists of length 1 to independent lists of length 1. Note that (v) is linearly independent if and only if v is non-zero! Hence if v is non-zero, (v) is linearly independent; thus (Tv) is linearly independent, so Tv is non-zero. Therefore the null space of T is $\{0\}$, which s what we wanted to prove.

3. *Let V be an* $\mathbf{F}$*-vector space such that* $\dim U \leq 4$ *for all finite-dimensional subspaces U of V. Prove that V is finite-dimensional and that its dimension is at most 4.*

This is kind of a silly problem (sorry). If V is *not* finite-dimensional, then there are independent lists $(v_1, \ldots, v_n)$ in V of arbitrary length. (We proved this in homework and discussed the proof in office hours a lot. This is a good place to cite a homework problem; alternatively, you could recapitulate the proof that there are such lists.) If $(v_1, \ldots, v_n)$ is linearly independent in V, its span is a subspace of V of dimension n. By the assumption of the problem, n can be at most 4, so we have a contradiction.

4. *Let $S : V \to W$ and $T : W \to V$ be linear maps between finite-dimensional* $\mathbf{F}$*-vector spaces. Suppose that TS is the identity map on V.*

(**a.**) *Prove that T is surjective (onto) and that S is injective (1-1).*

This has little to do with linear algebra: it's just a fact about composed functions. The hypothesis is that $T(S(v)) = v$ for all v in V. We see that the range of T is all of v because each v is T of something, namely v is $T(S(v))$. In a similar vein, if $S(v) = S(v')$, we get $v = v'$ by applying T to both sides of the equation $S(v) = S(v')$. Hence S is indeed 1-1.

(**b.**) *Show that we have* $\dim V \leq \dim W$.

We can use, for instance, the fact that the dimension of V is the sum of the nullity of S and the rank of S. Since S is 1-1, the nullity is 0; thus $\dim V = \operatorname{rank} S$. Since the range of S is a subspace of W, the dimension of the range is at most the dimension of W. In other words, as we discussed in class, $\operatorname{rank} S \leq \dim W$. We thus have the desired inequality $\dim V \leq \dim W$.