Math H110 Professor K. Ribet

Review problems for further study

1. Let A be an $n \times n$ matrix over a field F. If $A^2 = A$, show that A is similar to a diagonal matrix whose diagonal entries are all 0 or 1.

2. Let $T\colon V \to W$ be a linear transformation and let X be a subspace of W. Assume that V and W are finite-dimensional. Let $T^{-1}(X)$ be the inverse image of X under T, i.e., the set of vectors in V that map to X. Recall that $T^{-1}(X)$ is a subspace of V. Show that the dimension of this subspace is at least $\dim V - \dim W + \dim X$.

3. Let $T\colon V \to V$, where V is finite-dimensional over F. Let $f(t)$ be the characteristic polynomial of T. Show that $f(t)$ factors non-trivially over F if and only if there is a subspace W of V, other than $\{0\}$ and V, such that $T(W) \subseteq W$.

4. If A and B are $n \times n$ matrices that commute with each other, and if A and B are both diagonalizable, show that A and B are *simultaneously* diagonalizable.

5. Let $A \in \mathbf{M}_{n\times n}(\mathbf{C})$ be a matrix for which $\operatorname{tr} A^i = 0$ for $i = 1, 2, \ldots, n$. Show that A is *nilpotent*, i.e., that $A^k = 0$ for some $k \geq 1$.

6. Let p, q, r and s be polynomials over F of degree ≤ 3. If all four polynomials vanish at 1, are the polynomials necessarily linearly dependent over F? If all four polynomials have the value 1 at 0, are they necessarily linearly dependent over F?

7. Suppose that $A = (a_{ij})$ is a complex $n \times n$ matrix. Assume that a_{ij} is non-zero whenever $i = j + 1$ and that $a_{ij} = 0$ when $i \geq j + 2$. Show that A has exactly one Jordan block for each of its eigenvalues.

8. If A and B are $n \times n$ matrices, show that the two matrices AB and BA have the same eigenvalues. (If λ is an eigenvalue of one, it's an eigenvalue of the other.)

9. Let A be an $n \times n$ complex matrix with trace 0. Show that A is similar to a matrix whose diagonal entries are all 0.

10. Let B be a 3×3 matrix whose nullity is 2. For each statement i–iii, supply a proof of the statement or exhibit a counterexample: (i) The characteristic polynomial of B is divisible by t^2; (ii) The trace of B is an eigenvalue of B; (iii) The matrix B is diagonalizable.

These, too, come from `http://math.berkeley.edu/%7Edesouza/pb.html`