Math 110 — Professor K. A. Ribet

Midterm Exam — November 3, 2003

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

1. Let V be a vector space over a field F and let v be a vector in V. Let $T\colon V \to V$ be a linear transformation. Suppose that $T^m(v) = 0$ for some positive integer m but that $T^{m-1}(v)$ is non-zero. Show that the span of $\{\, v, T(v), \ldots, T^{m-1}(v)\,\}$ has dimension m.

We have to show that the vectors $v, T(v), \ldots, T^{m-1}(v)$ are linearly independent. Assume the contrary, i.e., that some non-trivial linear combination of these vectors vanishes. Take a linear combination with the fewest possible terms and write it in the form $0 = \sum_{i=0}^{r} a_i T^i(v)$ with $r \le m-1$. We must have $a_r \neq 0$ because otherwise we could re-write the sum with fewer terms. Also, some a_i with $i < r$ must be non-zero because otherwise the whole sum would consist of one term and we'd end up with $T^r(v) = 0$, which is contrary to assumption. We apply T^{m-r} to the linear combination and obtain $0 = \sum_{i=0}^{r} a_i T^{i+m-r}(v) = \sum_{i=0}^{r-1} a_i T^{i+m-r}(v)$, with the latter equality coming from the assumption $T^m(v) = 0$. We thus have a vanishing linear combination with fewer terms than the "minimal" one that we started with; this is a contradiction.

2. Let $T\colon V \to V$ be a linear map on a non-zero finite-dimensional vector space V over a field F. Suppose that the characteristic polynomial of T splits over F into a product of linear factors. Show that there is a basis B of V such that $[T]_{\mathrm{B}}$ is upper-triangular.

We prove the statement by induction on the dimension of V; it is trivial if $\dim V = 1$. The characteristic polynomial of T has a root, and thus T has an eigenvector v_1. The vectors that are multiples of v_1 form a 1-dimensional subspace W of V that is invariant under T (in the sense that $T(W) \subseteq W$). As we saw in the homework that was due on October 17, T induces a linear map

$U\colon V/W \to V/W$ whose characteristic polynomial divides that of T. By the induction hypothesis, there is a basis $\overline{v}_2, \dots, \overline{v}_n$ of V/W in which the matrix of U is upper-triangular. Here, we understand that we are choosing vectors $v_2, \dots, v_n$ of V and that the $\overline{v}_i$ are their images $v_i + W$ in V/W. The vectors $v_1, v_2, \dots, v_n$ form a basis of V in which T is upper-triangular.

3. Let V be an n-dimensional real or complex inner product space. Let $e_1, \dots, e_n$ be an orthonormal basis of V. Suppose that $T\colon V \to V$ is a linear transformation and let $T^*\colon V \to V$ be the adjoint of T. Show that $\displaystyle\sum_{j=1}^{n} \|T^*(e_j)\|^2 = \sum_{j=1}^{n} \|T(e_j)\|^2$. If $\|T^*v\| \leq \|Tv\|$ for all $v \in V$, show that $\|T^*v\| = \|Tv\|$ for all $v \in V$.

For the first part, we just compute. Say $T(e_j) = \displaystyle\sum_i a_{ij} e_i$ for each i. Then $\|T(e_j)\|^2 = \displaystyle\sum_i |a_{ij}|^2$, so that $\displaystyle\sum_{j=1}^{n} \|T(e_j)\|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$. The analogues of the a_{ij} for T^* are the $\overline{a}_{ji}$, but interchanging i and j and applying a complex conjugation does not change the double sum that we have just computed. Hence we indeed have $\displaystyle\sum_{j=1}^{n} \|T^*(e_j)\|^2 = \sum_{j=1}^{n} \|T(e_j)\|^2$. For the second part, we suppose that $\|T^*v\| < \|Tv\|$ for some $v \in V$. This vector must be non-zero; we can assume that it has norm 1 by dividing it by its norm. (This preserves the inequality that we started with.) Complete the singleton set $\{v\}$ to a basis $v = v_1, v_2, \dots, v_n$ of V and apply the Gram-Schmidt process to this basis to obtain an orthogonal basis of V. Divide the resulting vectors by their norms to obtain an orthonormal basis $e_1, \dots, e_n$ of V. The first element of this basis, e_1, is v. We now look at $\displaystyle\sum_{j=1}^{n} \|T^*(e_j)\|^2 = \sum_{j=1}^{n} \|T(e_j)\|^2$, comparing terms. For each j, the jth term on the left-hand side is less than or equal to the jth term on the right-hand side. On the other hand, the first term on the left is strictly less than the first term on the right. This is a contradiction: the sums cannot turn out to be equal. Our hypothesis $\|T^*v\| < \|Tv\|$ was therefore incorrect, so we are forced to conclude that $\|T^*v\| = \|Tv\|$ for all v.