Math H110 Professor K. Ribet

Homework assignment #9, due October 24

• My colleague Arthur Ogus pointed out that the textbook's definition of "characteristic polynomial" on page 248 is somewhat unsatisfying. Specifically, the book defines $f_A(t)$ as a determinant of a matrix whose entries are polynomials in t (of degree ≤ 1). As the authors note in a footnote, the characteristic polynomial is thus an element of the field $F(t)$ of rational functions (i.e., quotients of polynomials) with coefficients in F. First of all, how do we know that the characteristic polynomial is actually a polynomial, and not an expression like $\frac{t^2+2t+17}{t+17}$? Second, how do we know that $f_A(\lambda) = \det(A - \lambda I)$ when λ is an element of F?

• If A is an $n \times n$ matrix of complex numbers, we have defined two endomorphisms S and T on the space $\mathrm{Sym}_n(\mathbf{C})$ of complex $n \times n$ symmetric matrices. Specifically, $S(B)$ was defined as $AB + BA^{\mathrm{t}}$ and $T(B)$ was defined as ABA^{t}; here, A^{t} denotes the transpose of A. Note that S and T depend on A but I did not write S_A and T_A for S and T because I didn't want to introduce burdensome notation.

As we noted in class, $\mathrm{Sym}_n(\mathbf{C})$ has dimension $\frac{n(n+1)}{2}$. There is a natural basis for $\mathrm{Sym}_n(\mathbf{C})$ that is indexed by the set of subsets of $\{1, 2, \ldots, n\}$ that have 1 or 2 elements. Namely, for each $\{i, j\}$ with i and j distinct, we can consider the matrix e_{ij} that has a 1 in the ijth and jith places and 0 elsewhere. Similarly, for each i between 1 and n we can let e_i be the diagonal matrix with a 1 in the iith place and zeros elsewhere. The e_{ij} together with the e_i form a basis of $\mathrm{Sym}_n(\mathbf{C})$. If A is a diagonal matrix, calculate the effects of S and T on each of the basis elements. Find the eigenvalues for S and the eigenvalues for T. Compute the characteristic polynomials of S and T.

If you have the courage, do the same calculations for the two endomorphisms (that were called S and T) of the space H of $n \times n$ complex Hermitian matrices. Note that H is a real vector space of dimension n^2. (We exploited this space only when n is odd, but I believe that this assumption is not relevant for the present discussion.)

If you succeed in computing the characteristic polynomials of S and T (in either situation) when A is not necessarily diagonal, this is much better.