• §2.3, problem 13 and §2.6, problem 10.

• Suppose that $A$ is an $n \times m$ matrix and $B$ is an $m \times n$ matrix, so that the products $AB$ and $BA$ are both defined. (They are square matrices of size $n$ and $m$, respectively.) Prove that $\text{tr}(AB) = \text{tr}(BA)$, thus generalizing the first assertion of the problem.

• Suppose that the $F$-vector spaces $F^n$ and $F^m$ are isomorphic. Using theorems that we have proved in class, explain briefly why $n$ equals $m$. Now consider the following alternative argument:

To give an isomorphism from $F^n$ to $F^m$ is to give linear maps $T : F^n \to F^m$ and $U : F^m \to F^n$ so that the two composites $T \circ U$ and $U \circ T$ are the identity maps of $F^m$ and $F^n$. Equivalently, we have to find matrices $A$ and $B$ of dimensions $n \times m$ and $m \times n$ so that $AB$ and $BA$ are the identity matrices of sizes $n$ and $m$. The trace of $AB$ is $n$, while the trace of $BA$ is $m$; thus $n = m$.

Can we use this argument to show that the dimension of a vector space is well defined, or is it better to stick with the proof given in the book (replacement theorem and some easy further argument)?

• §2.4, problems 9, 20, 24

• Let $X$ be a subspace of a finite-dimensional $F$-vector space $V$. Let $V^*$ be the dual space of $V$ and define $X^\perp$ to be the subspace of $V^*$ consisting of those linear forms $\varphi : V \to F$ that vanish identically on $X$.

Recall the “canonical” map $\pi : V \to V/X$ that maps $v \in V$ to $v + X$. We obtain the linear map $\pi^* : (V/X)^* \to V^*$ by composing with $\pi$; a linear form $\varphi : V/X \to F$ maps to the linear form $\varphi \circ \pi : V \to F$. Show that $\pi^*$ is injective and that its image is $X^\perp$. Thus $X^\perp$ may be viewed as the dual of $V/X$.

• §2.6, problems 10 and 11