Math H110 Professor K. Ribet

Homework assignment #5, due September 26

• §2.3, problem 13 and §2.6, problem 10.

• Suppose that A is an $n \times m$ matrix and B is an $m \times n$ matrix, so that the products AB and BA are both defined. (They are square matrices of size n and m, respectively.) Prove that $\text{tr}(AB) = \text{tr}(BA)$, thus generalizing the first assertion of the problem.

• Suppose that the F-vector spaces F^n and F^m are isomorphic. Using theorems that we have proved in class, explain briefly why n equals m. Now consider the following alternative argument:

> *To give an isomorphism from F^n to F^m is to give linear maps $T : F^n \to F^m$ and $U : F^m \to F^n$ so that the two composites $T \circ U$ and $U \circ T$ are the identity maps of F^m and F^n. Equivalently, we have to find matrices A and B of dimensions $n \times m$ and $m \times n$ so that AB and BA are the identity matrices of sizes n and m. The trace of AB is n, while the trace of BA is m; thus $n = m$.*

Can we use this argument to show that the dimension of a vector space is well defined, or is it better to stick with the proof given in the book (replacement theorem and some easy further argument)?

• §2.4, problems 9, 20, 24

• Let X be a subspace of a finite-dimensional F-vector space V. Let V^* be the dual space of V and define $X^\perp$ to be the subspace of V^* consisting of those linear forms $\varphi : V \to F$ that vanish identically on X.

Recall the "canonical" map $\pi : V \to V/X$ that maps $v \in V$ to $v + X$. We obtain the linear map $\pi^t : (V/X)^* \to V^*$ by composing with π; a linear form $\varphi : V/X \to F$ maps to the linear form $\varphi \circ \pi : V \to F$. Show that π^t is injective and that its image is $X^\perp$. Thus $X^\perp$ may be viewed as the dual of V/X.

• §2.6, problems 10 and 11