Math H110 Professor K. Ribet

Homework assignment #14, due December 3

1. Exhibit a list of matrices over $\mathbf{R}$ that is as long as possible and so that the list has the following properties: (1) The characteristic polynomial of each matrix on the list is $(t-1)^5(t+1)$; (2) the minimal polynomial of each matrix is $(t-1)^2(t+1)$; (3) no two matrices on the list are similar to each other.

2. Suppose that M is a real 3×3 matrix such that M^3 is the identity matrix but M is not the identity matrix. Find the characteristic polynomial of M. Give an explicit example of an M satisfying the condition.

3. Determine the Jordan canonical form of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$.

4. Suppose that A is a real $n \times n$ matrix. Is it true that A must commute with its transpose? Suppose that the columns of A form an orthonormal set; is it true that the rows of A must also form an orthonormal set?

5. Suppose that A is an $n \times n$ matrix of real numbers. Let r be the rank of A. Let r' be the rank of the same matrix, considered as a matrix of complex numbers. Find a formula relating r' to r.

6. Let A be an $n \times n$ matrix of real numbers such that the sum of each column of A is the number 1. Show that there is a non-zero column vector x such that $Ax = x$.

7. Let T be a linear transformation on a finite-dimensional complex vector space V. We say that T is *completely reducible* if the following property holds: for each subspace W of V that is T-invariant (in the sense that $T(W) \subseteq W$), there is a T-invariant subspace W' of V so that $V = W \oplus W'$. Prove that T is completely reducible if and only if V has a basis of eigenvectors for T.

8. Let $\langle\, , \rangle$ be an inner product on a finite-dimensional complex vector space V. Let $T\colon V \to V$ be a linear transformation. Suppose that $\langle T(x), T(y)\rangle = 0$ for all $x, y \in V$ such that $\langle x, y\rangle = 0$. Prove that there is a unitary operator $S\colon V \to V$ and a complex number c so that $T = cS$.

9. Let A be a real $n\times n$ matrix such that $\det A$ is non-zero. Show that there is a polynomial $f(t)$ with real coefficients and degree $< n$ such that $A^{-1} = f(A)$.

10. Let A and B be $n \times n$ matrices over a field F. Suppose that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if they have the same rank.

11. Let $T\colon V \to V$ be a linear transformation on a finite-dimensional vector space. Let $n = \dim V$ and assume that $n \geq 2$. Suppose that $T^n = 0$ but that T^{n-1} is non-zero. Show that there is no linear transformation $U\colon V \to V$ such that $U^2 = T$.

12. Let V be the 9-dimensional real vector space $M_{3\times 3}(\mathbf{R})$. For each $A \in M_{3\times 3}(\mathbf{R})$, let $T_A\colon V \to V$ be the left-multiplication map $B \mapsto AB$. Suppose that A has determinant 32 and minimal polynomial $(t-4)(t-2)$. Find the trace of T_A.

These come from `http://math.berkeley.edu/%7Edesouza/pb.html`