Math 110 — Professor K. A. Ribet

Final Exam — December 16, 2003

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

1. Let A be an $n \times n$ matrix. Suppose that there is a non-zero row vector y such that $yA = y$. Prove that there is a non-zero column vector x such that $Ax = x$. (Here, A, x and y have entries in a field F.)

This is a restatement of problem 6 on HW #14. What is given is that 1 is an eigenvalue of the transpose of A. It follows that 1 is an eigenvalue of A; this gives the conclusion.

2. Let A and B be $n \times n$ matrices over a field F. Suppose that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if they have the same rank.

This is problem 10 in HW #14. If A and B are similar, then they certainly have the same rank. Indeed, we saw early on in the course that the rank of a matrix does not change if you multiply it on either side by an invertible matrix. The harder direction is the converse. If $T^2 = T$, where T is a linear operator on a vector space V, then we know well that V is the direct sum of the null space of T and the space of vectors that are fixed by T. (See, e.g., problem 17 on page 98 of the textbook.) The dimension of this latter space is clearly the rank of T. Choose a basis $v_1, \dots, v_r$ for the range of T and a basis $v_{r+1}, \dots, v_n$ for the null space of T. The matrix of T with respect to the basis $v_1, \dots, v_n$ is the direct sum of the $r \times r$ identity matrix and the $(n-r) \times (n-r)$ zero-matrix. Taking now $T = L_A$, we see that A is similar to a matrix that depends only on its rank. If A and B have the same rank, they are each similar to a common matrix, so they're similar to each other.

3. Suppose that $T\colon V \to V$ is a linear transformation on a finite-dimensional real inner product space. Let T^* be the adjoint of T. Show that V is the direct sum of the null space of T and the range of T^*.

The rank of T^* coincides with the rank of T for various reasons. For example, in matrix terms, this equality is the statement that a square matrix and its transpose have the same rank. Hence the dimensions of the null space of T and the range of T^* add up to the dimension of V. This necessary condition for V to be the indicated direct sum is a good sign! Also, it means that V is the direct sum of the two spaces if and only if V is the sum of the two spaces and that V is the direct sum of the two spaces if and only if the spaces have zero intersection in V. Let us prove the latter statement. Suppose that $T(v) = 0$ and that $v = T^*(w)$ for some w. We need to prove that $v = 0$. It is enough to show that $\langle v, v\rangle = 0$. But $\langle v, v\rangle = \langle v, T^*(w)\rangle = \langle T(v), w\rangle = \langle 0, w\rangle = 0$.

4. Let A be a symmetric real matrix whose square has trace 0. Show that $A = 0$.

Use the fact that A is similar to a diagonal matrix. If B is similar to A, then B has the same trace as A; also, $B = 0$ if and only if $A = 0$. Hence we can, and do, assume that A is a diagonal matrix. Say that the diagonal entries are $a_1, \ldots, a_n$. The hypothesis is that $\sum a_i^2 = 0$. Since the a_i are real numbers, they all must be 0. Hence $A = 0$.

5. Let $T\colon V \to W$ be a linear transformation between finite-dimensional vector spaces. Let X be a subspace of W. Let $T^{-1}(X)$ be the set of vectors in V that map to X. Show that $T^{-1}(X)$ is a subspace of V and that $\dim T^{-1}(X) \geq \dim V - \dim W + \dim X$.

This seems to be problem 2 of the "further review problems." As I write this answer, I have the impression that the problem is harder than I thought, but perhaps there's an easier way to say what I'm about to explain. Let Y be the range of T, so that X and Y are both subspaces of W. A third subspace is $X \cap Y$. Consider the quotient space W/X and the natural map $\iota\colon Y \to W/X$ that sends $y \in Y$ to $y + X$. The null space of this map is $Y \cap X$. Hence $\dim Y = \dim(Y \cap X) + \operatorname{rank}(\iota) \leq \dim(Y \cap X) + \dim(W/X) = \dim(Y \cap X) + \dim W - \dim X$. Now let U be the restriction of T to $T^{-1}(X)$. Since $T^{-1}(X)$ contains the null space of T, the nullity of U is the same thing as the nullity of T. The range of U is $Y \cap X$. We have $\dim T^{-1}(X) = \operatorname{nullity}(T) + \dim(Y \cap X) \geq \operatorname{nullity}(T) + \dim X + \dim Y - \dim W = \dim V + \dim X - \dim W$, where we have used the equality $\dim V = \operatorname{nullity}(T) + \operatorname{rank}(T) = \operatorname{nullity}(T) + \dim Y$.

6. Suppose that V is a real finite-dimensional inner product space and that $T\colon V \to V$ is a linear transformation with the property that $\langle T(x), T(y)\rangle = 0$

whenever x and y are elements of V such that $\langle x, y\rangle = 0$. Assume that there is a non-zero $v \in V$ for which $\|T(v)\| = \|v\|$. Show that T is orthogonal.

This is a slightly friendlier version of problem 8 on HW #14. After scaling v, we may assume that $\|v\| = 1$. Complete v to a basis of V and then apply the Gram-Schmidt process. We emerge with an orthonormal basis $e_1, \dots e_n$ of V with $v = e_1$. Let i be greater than 1 and let $w = e_i$. Then $\langle v+w, v+w\rangle = \langle v, v\rangle + \langle w, w\rangle = 2$ because $v \perp w$. Similarly, $\langle T(v+w), T(v+w)\rangle = \|T(v)\|^2 + \|T(w)\|^2$ because $T(v) \perp T(w)$ by hypothesis. It follows that $\|T(w)\|^2 = 1$ because we knew already that $\|T(v)\|^2$ was 1. In other words, we have $\|T(e_i)\| = 1$ for all i; equivalently $\langle T(e_i), T(e_j)\rangle = \delta_{ij}$ for all i and j. It follows by linearity that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for $x, y \in V$. Thus T is orthogonal.

7. Let T be a nilpotent operator on a finite-dimensional complex vector space. Using the table

i	0	1	2	3	4	5	$\cdots$
nullity(T^i)	0	4	7	9	10	10	$\cdots$,

find the Jordan canonical form for T.

I think that there are four blocks, all of which pertain to the eigenvalue 0, which is the sole eigenvalue here. There's one block of length 1, one of length 2, one of length 3 and one of length 4. The vector space has dimension 10.

8. Let F be a finite field; write q for the number of elements of F. Let V be an n-dimensional vector space over F. Compute, in terms of n and q, the number of 1-dimensional subspaces of V and the number of linear transformations $V \to V$ that have rank 1.

The number of non-zero vectors in V is $q^n - 1$. Each such vector generates a 1-dimensional subspace. On the other hand, the non-zero multiples of a vector v generate the same subspace as v. Each non-zero vector has $q - 1$ multiples. Hence the number of 1-dimensional subspaces is $(q^n - 1)/(q - 1)$. If W is a 1-dimensional subspace of V, the number of linear maps $V \to W$ is q^n. Among these maps is the zero-map, which is not of rank 1. The number of rank-1 maps $V \to V$ whose range is W is therefore $q^n - 1$. All told, there are $(q^n - 1)^2/(q - 1)$ linear transformations $V \to V$ that have rank 1.