Math 110

PROFESSOR KENNETH A. RIBET

Final Exam

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12:30–3:30 PM

The scalar field F will be the field of real numbers unless otherwise specified.

> Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in. Don't trust staples to keep your papers together. Explain your answers as is customary and appropriate. Your paper is your ambassador when it is graded.

Disclaimer: These solutions were written by Ken Ribet. As usual, sorry if they're a bit terse and apologies also if I messed something up. If you see an error, send me e-mail and I'll post an updated document.

1. *Let $T : V \to V$ be a linear transformation. Suppose that all non-zero elements of V are eigenvectors for T. Show that T is a scalar multiple of the identity map, i.e., that there is a $\lambda \in \mathbb{R}$ such that $T(v) = \lambda v$ for all $v \in V$.*

We can and do assume that V is non-zero. Choose v non-zero in V and let λ be the eigenvalue for v. We must show that $Tw = \lambda w$ for all $w \in W$. This is clear if w is a multiple of v. If not, w and v are linearly independent, so that $w + v$ is non-zero, in particular. In this case, let μ be the eigenvalue of w and let a be the eigenvalue of $w + v$. Then $a(w + v) = T(w + v) = Tw + Tv = \mu w + \lambda v$, and so $(a - \mu)w = (\lambda - a)v$. Because v and w are linearly independent, we get $a = \mu$ and $a = \lambda$. Hence $\mu = \lambda$.

2. *Let V be a 7-dimensional vector space over $\mathbb{R}$. Consider linear transformations $T : V \to V$ that satisfy $T^2 = 0$. What are the possible values of* nullity(T)*? (Be sure to justify your answer: don't just reply with a list of numbers.)*

I believe that 4, 5, 6, and 7 are the possible values. The condition $T^2 = 0$ may be rephrased as the inclusion $R(T) \subseteq N(T)$. Since the dimensions of $R(T)$ and $N(T)$ need to sum to 7, we must have $\text{nullity}(T) \geq 4$. Conversely, for desired value of $\text{nullity}(T)$ between 4 and 7, we can fabricate a T that gives this value. For example, to have $\text{nullity}(T) = 5$, we define T on the standard basis vectors $e_1, \ldots, e_7$ by having $T(e_1) = \cdots T(e_5) = 0$, $T(e_6) = e_1$ and $T(e_7) = e_2$. This works: the rank is at least 2 because the range contains the space spanned by e_1 and e_2 while the nullity is at least 5 because the first 5 basis vectors are sent to 0 by T.

3. *Let A be a real $n \times n$ matrix and let A^t be its transpose. Prove that L_A and L_{A^tA} have the same null space. In other words, for x in $\mathbb{R}^n$, regarded as a column vector, show that $A^tAx = 0$ if and only if $Ax = 0$. Prove also that the linear transformations L_{A^t} and L_{A^tA} have the same range. (It may help to introduce an inner product on $\mathbb{R}^n$.)*

Use the standard inner product on $\mathbb{R}^n$. If Ax is non-zero, then the inner product $\langle Ax, Ax\rangle$ is positive. Re-write as $\langle x, A^tAx\rangle$ to see that A^tAx is non-zero. The statement about the null space is what we have just proved. the one about ranges follows by dimension considerations, since one range is a priori contained in the other.

4. *Let $T : V \to V$ be a linear transformation. Suppose that $v_1, v_2, \ldots, v_k \in V$ are eigenvectors of T that correspond to distinct eigenvalues. Assume that W is a T-invariant subspace of V that contains the vector $v_1 + v_2 + \cdots + v_k$. Show that W contains each of $v_1, v_2, \ldots, v_k$.*

This is a recycled quiz problem. See Tom's web page for a solution.

Let $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ be a real square matrix of size n. Let $v_1, \ldots, v_n$ be the rows of A. Assume that $\langle v_i, v_j\rangle = \delta_{ij}$ for $1 \leq i, j \leq n$; here, $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$ and δ_{ij} is the Kronecker delta. Show that the corresponding statement holds for the columns of A, i.e., that the inner product of the ith and the jth columns of A is δ_{ij}.

I discussed this in class a week ago. After you write down what's involved, you see that you need to know only that $BA = I$ if $AB = I$ when A and B are square matrices.

5. *Let $A = (a_{ij})$ be an $n \times n$ real matrix with the following properties:*

(1) *the diagonal entries are positive;*
(2) *the non-diagonal entries are negative;*
(3) *the sum of the entries in each row is positive.*

Suppose, for each $i = 1, \ldots, n$, that we have

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0,$$

where the x_i are real numbers. Show that all x_i are 0. [Assume the contrary and let i be such that x_i is at least as big in absolute value as the other x_k. Consider the ith equation.] Prove that $\det A$ is non-zero.

We assume the contrary and let i be such that x_i is at least as big in absolute value as the other x_k. After changing the sign of the x_k, we can and do assume that x_i is positive. Then

$$a_{ii}x_i = \sum_{j\neq i}(-a_{ij})x_j \leq \sum_{j\neq i}(-a_{ij})x_i = x_i\sum_{j\neq i}(-a_{ij}).$$

Dividing by x_i, we get $a_{ii} \leq -\sum_{j\neq i} a_{ij}$, which is contrary to the hypothesis that the sum of the entries in the ith row is positive. This business with the x_k shows that the null space of L_A is $\{0\}$. We conclude that A is invertible, so that its determinant is non-zero.

6. *Decide whether or not each of the following real matrices is diagonalizable over the field of real numbers:*

$$\begin{pmatrix} 10 & 9 \\ 0 & 10 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 9 & 1 \\ 2 & 9 & 9 & 6 \\ 9 & 9 & 3 & 4 \\ 1 & 6 & 4 & -1 \end{pmatrix}$$

Choose one of the above three matrices—call it A. Exhibit a matrix Q of the same size as A such that $Q^{-1}AQ$ is diagonal.

The first matrix is a quintessential example of a non-diagonalizable matrix, as discussed in class. Its only eigenvalue is 10. If it were diagonalizable, it would be 10 times the identity matrix. The third matrix is diagonalizable because it is symmetric; all I want here is that you quote the theorem to the effect that symmetric matrices can be diagonalized over $\mathbb{R}$. The middle matrix is diagonalizable because $\mathbb{R}^3$ has a basis of eigenvectors for it: there's some eigenvector with eigenvalue 3, and we see that e_1 and e_2 are eigenvectors with eigenvalue 1. For this matrix, we can find Q by finding an eigenvector with eigenvalue 3. Such a vector has to have a non-zero third coefficient, since otherwise it would have eigenvalue 1. We can scale it so that it's of the form $(a, b, 1)$. Apply the matrix to this vector and see what condition on a and b we have if we want the vector to be turned into 3 times itself. The answer is that we need $a = b = 1$. And, indeed, you can check instantly that $(1, 1, 1)$ is an eigenvector with eigenvalue 3. We take Q here to be the matrix whose colums are the three eigenvectors, namely $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.