Methods of Mathematics

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Some perspective on this week's homework

Check out the Statement on p-values that was released yesterday by the American Statistical Association:

- P-values can indicate how incompatible the data are with a specified statistical model.
- P-values do not measure the probability that the studied hypothesis is true, or the probability that the data were produced by random chance alone.
- . . .
- By itself, a p-value does not provide a good measure of evidence regarding a model or hypothesis.

The full statement is available to you on bCourses.

Office hours

Monday 2:10–3:10 and Thursday 10:30–11:30 in Evans

Tuesday 10:30–noon at the SLC

Sequences

Sequences are numerical functions that are defined on the set of non-negative integers (or sometimes the set of positive integers). A typical letter to denote a sequence is "a," but we tend to write a_n instead of $a(n)$.

Examples: $a(n) = n^2$; $a_n =$ the nth prime number (for $n \geq 1$); $a(n) = n!$; ....

Example: $a_n = \binom{n}{10}$ for $n \geq 0$. The sequence begins with ten 0s and then continues 1, 11, 66, 286, 1001, 3003, 8008, 19448, 43758, 92378, 184756, 352716, 646646, and so on.

If you like sequences, you'll *love* the on-line encyclopedia of integer sequences.

See especially the OEIS discussion of sequences associated with the life sciences.

If you're from New York, check out `https://oeis.org/A000053`.

Recursive definitions

The sequence $a_n = n!$ can be defined by the rules

- $a_0 = 1$,
- $a_n = na_{n-1}$ for $n \geq 1$.

Here's an analogous sequence s_n that's made with addition:

- $s_0 = 0$,
- $s_n = n + s_{n-1}$ for $n \geq 1$.

Then s_n is the sum of the first n positive integers, just as $n!$ is the product of the first n positive integers. Because we are masters of mathematical induction, we know:

$$s_n = \frac{n(n+1)}{2} \text{ for } n \geq 0.$$

Fibonacci numbers

The Fibonacci numbers are:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

or sometimes just

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

The key points are that the first few are as depicted and that each number (starting with the third) is the sum of the two numbers to its left.

The sequence that includes 0 is the one in Rosen's discrete math book. It's described by Wikipedia as the more "modern" version. If the numbers in the modern sequence are F_n ($n \geq 0$), then the definition reads:

- $F_0 = 0$, $F_1 = 1$ (initial conditions);
- $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Our textbook prefers the second sequence (the one without 0) and refers to it as a_n. Then

$$a_n = F_{n+1}$$

for $n \geq 0$.

A natural question is then: “If you refer to *the* Fibonacci numbers, which of the two sequences are you talking about?" For this course, let’s go with the “textbook” definition and use the sequence a_n. Just be aware that the authors of the textbook are in a minority. (For Math 10B in 2017, the authors will probably change to the modern definition.)

A midterm question

> *. . . let $R(n)$ be the number of bit strings of length n that do not contain* `00`. *We have $R(0) = 1$, $R(1) = 2$, $R(2) = 3$,. . . .*

These numbers are the Fibonacci numbers except that the index has been shifted. Specifically, $R(n) = a_{n+1} = F(n+2)$ (if I'm not mistaken). You essentially proved this when you worked MT#1.

In how many ways can I walk up a flight of n steps, taking the steps one or two at a time?

Translation: In how many ways can we write n as a sum of 1s and 2s, where the order of the summands counts?

For example, for $n = 5$, I can write 5 as the sum of five 1s; the sum of 2 and 3 1s; and the sum of two 2s and one 1. Because order counts, the answer to our question is $1 + 4 + 3 = 8$, which is F_6.

I can write $4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2$. The number of ways to climb a flight of four stairs is then $1 + 3 + 1 = 5 = F_5$.

The answer for n stairs has got to be $a_n = F_{n+1}$, right? Can you prove this??

Brady numbers

The Brady numbers B_n are defined by the Fibonacci recursion

$$B_n = B_{n-1} + B_{n-2}$$

and the initial values $B(1) = 2308$, $B(2) = 4261$. (Apparently, there is no $B(0)$.) The initial values 2308 and 4261 are visible on many of Brady's t-shirts, but not on all of them.

Fibonacci numbers on google plus

While preparing this lecture, I found a reference (on Facebook) to a google plus post by Allen Knutson about this assertion:

> *If a Fibonacci number is a multiple of 9, it's a multiple of 8.*

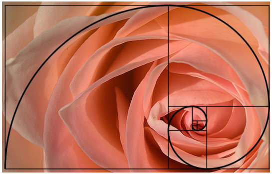
Fibonacci numbers have been studied intensively. There's even a research journal devoted to Fibonacci issues.

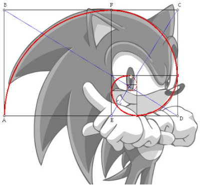
Fibonacci numbers in nature

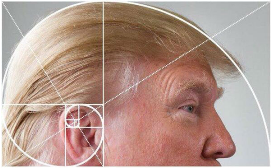
Many books on discrete math include photos of snail shells and plants of various kinds. My colleague Vivek Shende presented a fine summary slide in the Math 54 lecture that he gave last Thursday:

The Fibonacci numbers

This spiral can be seen in nature...







The formula linking Fibonacci numbers shows that these numbers satisfy a *linear recursion relation* (of order 2). The general setup (order 2) would be a situation where

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_d a_{n-d}$$

for $n \geq d$; here, the αs are constants (both equal to 1 in the Fibonacci example). To determine the sequence, we need to give supply also the d initial values

$$a_0, a_1, \ldots, a_{d-1}.$$

In principle, the sequence a_n is then given by a closed-form formula that involves the roots of the *characteristic polynomial*

$$\lambda^n - (\alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_d).$$

To illustrate, let's begin with a fairly simple quadratic polynomial, say

$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2);$$

its roots are 3, -2. The associated recursion relation is

$$a_n = a_{n-1} + 6a_{n-2}.$$

We think of this equation as analogous to a differential equation and describe a sequence a_n as a *solution* to the equation if it satisfies the relation. A key observation is that

$$a_n := 3^n, \quad a_n = (-2)^n$$

are two solutions to the relation. Another is that multiples of solutions are again solutions and that the sum of two solutions is again a solution. (That's why the word *linear* is used.)

A consequence of all this is that if C_1 and C_2 are fixed numbers, the sequence

$$C_1 3^n + C_2(-2)^n$$

satisfies the given recursion

$$a_n = a_{n-1} + 6a_{n-2}.$$

For ease of notation, it might be best to say the same thing this way: if C_1 and C_2 are fixed numbers, the sequence

$$b_n := C_1 3^n + C_2(-2)^n$$

satisfies

$$b_n = b_{n-1} + 6b_{n-2}.$$

Now suppose we are told that $a_0 = 2$, $a_1 = 1$, and $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$. Suppose further that we seek to find a formula for the numbers a_n. Our aim will be to find numbers C_1 and C_2 so that $a_n = C_1 3^n + C_2(-2)^n$ for all n. If the formula is correct, then in particular $2 = a_0 = C_1 + C_2$ and $1 = a_1 = 3C_1 - 2C_2$. Using high school algebra (or otherwise) we see that $C_1 = C_2 = 1$. Thus *if*

$$a_n = C_1 3^n + C_2(-2)^n$$

for all n, *then*

$$a_n = 3^n + (-2)^n$$

for all n. The whole point is that we can show that indeed

$$a_n = 3^n + (-2)^n \text{ for all } n \geq 0.$$

We don't have to use rocket science to do this. Let $b_n := C_1 3^n + C_2(-2)^n = 3^n + (-2)^n$. We have rigged the situation so that

$$a_0 = b_0, \quad a_1 = b_1.$$

Moreover, for $n \geq 2$ we have

- $a_n = a_{n-1} + 6a_{n-2}$,
- $b_n = b_{n-1} + 6b_{n-2}$.

Since the *b*s match the *a*s at the start of the sequence and are defined recursively by the same formula as the *a*s, the *b*s and the *a*s will be the same for all values of the index n.

Brady again

Solve the equation

$$B_n = B_{n-1} + B_{n-2}, n \geq 3,$$

given the initial conditions $B(1) = 2308$, $B(2) = 4261$.

The characteristic equation $\lambda^2 - \lambda - 1$ has the two roots $\frac{1 \pm \sqrt{5}}{2} = 1.61803398874989, -0.618033988749895$.

The positive root 1.61803398874989 is the famous golden ratio,

which might be the only quadratic irrational number to have a town square named after it. (Photo taken by me in July, 2013.)

The analysis that we applied to the sequence a_n a few minutes ago tells us that there's a formula

$$B_n = C_1(\frac{1+\sqrt{5}}{2})^n + C_2(\frac{1-\sqrt{5}}{2})^n$$

that's valid for all $n \geq 1$. The only task ahead of us is to solve the equations

$$C_1(\frac{1+\sqrt{5}}{2}) + C_2(\frac{1-\sqrt{5}}{2}) = 2308$$

and

$$C_1(\frac{1+\sqrt{5}}{2})^2 + C_2(\frac{1-\sqrt{5}}{2})^2 = 4261.$$

The second equation may be rewritten

$$C_1(1 + \frac{1+\sqrt{5}}{2}) + C_2(1 + \frac{1-\sqrt{5}}{2}) = 4261.$$

Subtracting the first equation from it gives $C_1 + C_2 = 1953$. Fiddling with the first equation, I got

$$1953 + (C_1 - C_2)\sqrt{5} = 4616 \Longrightarrow C_1 - C_2 = \frac{1}{\sqrt{5}}2663.$$

Using the equations for $C_1 + C_2$ and $C_1 - C_2$, we can find the values

$$C_1 = \frac{1953}{2} + \frac{2663}{2\sqrt{5}}$$

and

$$C_2 = \frac{1953}{2} - \frac{2663}{2\sqrt{5}}.$$

In summary, B_n is given by the appealing expression

$$\left(\frac{1953}{2} + \frac{2663}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1953}{2} - \frac{2663}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for all $n \geq 1$. I wonder whether Brady has seen this formula before?

Formula for the Fibonacci numbers

There's a similar formula for the Fibonacci numbers (cf. page 24 of the Dynamics "textbook"). The numbers C_1 and C_2 are just $\pm\frac{1}{\sqrt{5}}$:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Inhomogeneous

Suppose we were confronted with a recursion like

$$a_n = a_{n-1} + 6a_{n-2} + f(n),$$

where $f(n)$ is a function of n; $f(n)$ might be 10^n, or perhaps a constant. Then we can solve this equation (giving a general formula for a_n) if we can find (by guessing, or otherwise) a single solution.

For example, consider

$$a_n = a_{n-1} + 6a_{n-2} + 10.$$

A fairly obvious particular solution is the constant function $a_n = -6/10 = -3/5$. The general solution would be $a_n = C_1 3^n + C_2(-2)^n - 3/5$, and we'd use the initial conditions (i.e., the values of a_0 and a_1) to find C_1 and C_2.

For example, if we were given $a_n = a_{n-1} + 6a_{n-2} + 10$ plus the initial conditions $a_0 = 0$, $a_1 = -4/5$, then we'd find (I hope!) that

$$a_n = \frac{1}{5}3^n + \frac{2}{5}(-2)^n - \frac{3}{5}.$$