Methods of Mathematics

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Math 10B
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Breakfasts

The next breakfast will be on Thursday, April 7 at 8:30AM. To sign up, send me email!

There's a "pop-up" lunch on Friday, April 8 (next week) at the Faculty Club at 12:30PM. All are welcome.

Midterm #2

The last midterm exam will be on Thursday, March 31.

The exam focuses on material discussed since February 9:

- Independence
- Random variables
- Discrete distributions
- Expected value
- Variance
- Sampling
- Estimation
- Hypothesis testing
- χ^2
- Recursion equations
- Differential equations, especially first order linear ODEs
- Partial fractions

There will be 5–7 questions, and the likely total point value will be 35 (same as for midterm #1).

You can bring in one two-sided page of notes ($8\frac{1}{2} \times 11$ inches).

No devices.

Write in complete English sentences. Write carefully and clearly.

Because of Gradescope, be as clear as possible as to the location of your answers. (There are two sheets of scratch paper as before.)

In lecture on Tuesday, could we go over questions similar to 6 and 7 on the homework with irreducible quadratics?

Sure! How about

$$\int \frac{x^3 + 21x^2 + 43x - 25}{x^4 + 4x^3 - 6x^2 - 20x - 75}\,dx?$$

This is too hard as presented, but the problem is OK if the denominator is factored into the product of two quadratics:

$$\int \frac{x^3 + 21x^2 + 43x - 25}{(x^2 + 2x - 15)(x^2 + 2x + 5)}\,dx.$$

To do the integral, here's what we have to do:

- figure out the general form of the partial fraction decomposition;
- solve for the (four) constants in the decomposition;
- integrate the partial fractions.

There's an immediate trap: the quadratic $x^2 + 2x - 15$ is *not* irreducible over the field of real numbers because its discriminant $2^2 - 4(-15) = 64$ is positive. The fact that the discriminant is a perfect square tells us that the quadratic actually has integer roots. We probably could have factored it by inspection:

$$x^2 + 2x + 15 = (x + 5)(x - 3).$$

Thus the denominator should be viewed as

$$(x+5)(x-3)(x^2+2x+5).$$

The partial fraction decomposition is then

$$\frac{x^3+21x^2+43x-25}{(x+5)(x-3)(x^2+2x+5)} = \frac{A}{x+5} + \frac{B}{x-3} + \frac{Cx+D}{x^2+2x+5}.$$

To solve for A, B, C and D, we multiply both sides of the equation by $(x+5)(x-3)(x^2+2x+5)$, obtaining:

$$\begin{aligned} &x^3+21x^2+43x-25 \\ &\quad = A(x-3)(x^2+2x+5) + B(x+5)(x^2+2x+5) \\ &\qquad + (Cx+D)(x+5)(x-3). \end{aligned}$$

The next step is to find out what we can by selecting special values for x. Good initial choices are $x = 3$, $x = -5$. When $x = 3$, we get $320 = B(8)(20)$, so $B = 2$. Similarly, if $x = -5$. then we obtain $160 = A(-8)(20)$, so $A = -1$. Now put $x = 0$:

$$-25 = A(-3)(5) + B(25) + D(-15) = 15 + 50 - 15D,$$

so $-90 = -15D$ and $D = 6$.

Finally we look at the coefficients of x^3 on the two sides of the equation:

$$1 = A + B + C = -1 + 2 + C;$$

this gives $C = 0$.

As far as integrating does, we have to integrate $\frac{1}{x+5}$ and $\frac{1}{x-3}$, but this is OK because both are known logs.

What about $\int \frac{6}{x^2+2x+5}\,dx$?

To compute the integral, we need to complete the square. This is relatively easy because $x^2+2x+1=(x+1)^2$, so $x^2+2x+5=(x+1)^2+2^2$. If $u=x+1$ and $a=2$, we have the famous expression u^2+a^2. Since $du=dx$ in this case, the integral to be computed is

$$6\int \frac{1}{u^2+a^2}\,du,$$

which is $\frac{6}{a}\arctan\frac{u}{a}+C=6\arctan(\frac{x+1}{2})+C$.

How would we proceed if we had instead to integrate

$$\int \frac{6x}{x^2 + 2x + 5}\,dx?$$

We'd write $x = u - 1$ and get

$$\int \frac{6u - 6}{u^2 + a^2}\,du.$$

The new element is

$$\int \frac{u}{u^2 + a^2}\,du,$$

which is a log.

"Tomorrow, could we also go over how to solve inhomogeneous recursion equations? For example, in the lecture slides (03/08), I am not entirely sure how to solve $a_n = a_{n-1} + 6a_{n-2} + 10$ to have a particular solution $a_n = -3/5$."

The first step is to solve the corresponding homogeneous equation; in this case, that equation is

$$a_n = a_{n-1} + 6a_{n-2}.$$

The characteristic polynomial is $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$. The roots of the polynomial are 3, -2, so the general solution is

$$a_n = C3^n + D(-2)^n,$$

where C and D are constants. (See the March 8 slides.)

The second step is to find *one* solution to $a_n = a_{n-1} + 6a_{n-2} + 10$. We can guess that we can take a_n to be a constant sequence, say $a_n = E$ (where E is a number). Then E needs to satisfy $E = E + 6E + 10$, so we get $E = -5/3$. (The value is not $3/5$. The correction was made on `piazza` about a week ago.)

The final step is to add together the two parts: the general solution to the inhomogeneous equation is

$$a_n = C3^n + D(-2)^n - 5/3.$$

Why this works was discussed a bit on `piazza`. Note that C and D are determined by initial conditions if they are given in the problem.

"Could we also do a sample mixing problem (similar, or the same as problem #13 from the homework)?"

A tank is used in hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of the dye in the tank reaches 1% of its original value.

The tank constantly has 200L of liquid in it. Let $y(t)$ be the amount of lye in the tank. The differential equation that we see is that $y'(t) = -0.01y(t)$: at any given moment, we are losing lye at the rate of 0.01 times the amount of lye that we have because liquid is flowing out at the rate of 0.01 times the amount of liquid in the tank (which is constant). (The rate is measured in g/min.)

The solution to the DE is $y(t) = y(0)e^{-0.01t}$, so we need to find t so that $e^{-0.01t} = 0.01$. We see that $t = 100\ln(100) \approx 460.52$ (minutes).

"In geometric distribution when does $E[X] = 1/p$ and when does $E[X] = (1-p)/p$?"

This question has been asked on `piazza` more than once. The two quantities differ by 1 and they pertain to two different random variables—which differ by 1.

One random variable counts the number of failures (tails) *before* the first success (heads).

The other counts the *total* number of trials to get the first success, including the successful trial.

Can someone explain bias? and how it affects the stat?

Presumably, the reference is to the slides on "Biased and unbiased estimators" on the section on Statistical Tests: Sampling and estimation.

I understand this subject at the $\sim 95\%$ confidence level. Let's imagine that we are trying to estimate the mean and variance of a random variable X on a probability space. The space can be really simple like $\{H, T\}$, and the random variable can take the values 0 on T and 1 on H. The true expected value or mean of this variable is the probability that H comes up on a given coin toss.

If we do lots of tosses and evaluate X repeatedly, we get samples x_i of X, say $i = 1, \ldots, n$. An estimate for the mean of X is then $\bar{x}$, the average of the x_i. One would think that the best estimate for the variance of X is the average value of $(X - \bar{x})^2$, which would be obtained by dividing $\sum(x_i - \bar{x})^2$ by n. However, a computation alluded to around p. 138 of the discrete probability "textbook" gives the information that it's more correct to divide by $n - 1$.

Let's explore this with $n = 2$. There is a random variable X whose mean and variance we don't know. When we sample to explore X, we are creating copies $X_1, X_2, \ldots$ of X; with $n - 2$, we have only X_1 and X_2. We estimate the mean of X with $\overline{X} = (X_1 + X_2)/2$. The claim is that the expected value of

$$(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2$$

is σ^2, the variance of X. Now the two squares in the displayed formula are equal; they're $\frac{1}{4}(X_1 - X_2)^2$. Hence the expected value of the sum of the squares is half the expected value of $(X_1 - X_2)^2$.

Because X_1 and X_2 are independent, this expected value is

$$E[X_2^2] + E[X_1^2] - 2E[X_1]E[X_2],$$

which is $E[X_2^2] + E[X_1^2] - E[X_1]^2 - E[X_2^2] = 2\sigma^2$.

Check out the new video on Bayes' Rule.