Methods of Mathematics

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Math 10B
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US News & World Report rankings:

Graduate School Search - Math Programs

School	Program Rank
Massachusetts Institute of Technology - Department of Mathematics Cambridge, MA	#1 Tie
Princeton University - Department of Mathematics Princeton, NJ	#1 Tie
Harvard University - Department of Mathematics Cambridge, MA	#3 Tie
University of California--Berkeley - Department of Mathematics Berkeley, CA	#3 Tie
Stanford University - Department of Mathematics Stanford, CA	#5 Tie

When it comes to math, we’ve got the Axe!

More on partial fractions

On Tuesday, we established the decomposition

$$\frac{5x^2+3x+3}{(x^2+x+1)(x+1)} = \frac{-2}{x^2+x+1} + \frac{5}{x+1}.$$

I ended by pointing out that my slides had other examples of partial fraction decomposition.

The first involves the decomposition of

$$\frac{6x^2 + 22x + 18}{(x+1)(x+2)(x+3)}$$

as a sum of partial fractions. We write

$$\frac{6x^2 + 22x + 18}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

and multiply to obtain

$$6x^2+22x+18 = A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2).$$

If we set $x = -1$, we find $2 = A \cdot 1 \cdot 2$, so $A = 1$. We find similarly that $B = 2$ and $C = 3$ (I hope!).

In the second example, we wanted to find A, B and C such that

$$\frac{9x^2 - 4x - 16}{(x^2 - x - 1)(x + 3)} = \frac{Ax + B}{x^2 - x - 1} + \frac{C}{x + 3}.$$

The first operation is to multiply so as to get

$$9x^2 - 4x - 16 = (Ax + B)(x + 3) + C(x^2 - x - 1).$$

Put $x = -3$ to obtain $77 = 11C$, so that $C = 7$. Then

$$9x^2 - 4x - 16 = (Ax + B)(x + 3) + 7(x^2 - x - 1).$$

Compare coefficients of x^2 to get $9 = A + 7$, so $A = 2$. Try $x = 0$; this gives $-16 = 3B - 7$, so $B = -3$.

Whoops! As we saw in lecture, $x^2 - x - 1$ is *not* an irreducible quadratic. The discriminant of this polynomial is 5, which happens not to be a negative number. (The roots of the polynomial are the Golden Ratio and the negative reciprocal of the Golden Ration.) In class, we changed the problem so that the rational function to be separated was

$$\frac{9x^2 - 4x - 16}{\left(x^2 - x + 1\right)(x + 3)}.$$

We started doing the computation but realized that the constants A, B and C involved fractions (yikes!). In particular, we found $C = 77/13$.

According to Sage, the full partial fraction expansion is

$$\frac{5\,(8\,x - 19)}{13\,\left(x^2 - x + 1\right)} + \frac{77}{13\,(x + 3)}.$$

The general rule is that each term like $(x+a)^n$ in the denominator corresponds to n partial fractions:

$$\frac{A_1}{x+a}, \frac{A_2}{(x+a)^2}, \dots, \frac{A_n}{(x+a)^n}.$$

Also, each term $(x^2+rx+s)^m$ in the denominator corresponds to the m partial fractions

$$\frac{B_i x + C_i}{(x^2+rx+s)^i}, \quad i = 1, \dots, m.$$

If the denominator has degree D, there are D different constants to be determined. We can find them by multiplying everything out and comparing coefficients of the various powers of x.

The following problem is from *Higher Algebra* by H. H. Hall and S. R. Knight, 1887: decompose into partial fractions

$$\frac{5x^3 + 6x^2 + 5x}{(x^2 - 1)(x + 1)^3}.$$

There is apparently a sketchy deliberate attempt to mislead the reader. It's important to realize that $x^2 - 1$ factors and that the denominator needs to be written $(x - 1)(x + 1)^4$. There are five partial fractions, which we can write $\frac{A}{x - 1}, \frac{B}{x + 1}, \ldots, \frac{E}{(x + 1)^4}$.

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Sage tells me that the answer is

$$-\frac{1}{x+1}+\frac{1}{x-1}+\frac{3}{(x+1)^2}-\frac{3}{(x+1)^3}+\frac{2}{(x+1)^4}.$$

This means: $A=1$, $B=-1$, $C=3$, $D=-3$, $E=2$.

To find this by hand, we might multiply both sides of the equation by the denominator $(x^2-1)(x+1)^3$, obtaining

$$\begin{aligned} 5x^3 &+ 6x^2 + 5x \\ &= A(x+1)^4 + B(x+1)^3(x-1) + C(x+1)^2(x-1) + \cdots \\ &= (A+B)x^4 + (4A+2B+C)x^3 + (6A+C+D)x^2 + \cdots . \end{aligned}$$

Comparing coefficients leads to the five equations

$$\begin{aligned} 0 &= A + B \\ 5 &= 4A + 2B + C \\ 6 &= 6A + C + D \\ 5 &= 4A - 2B - C + E \\ 0 &= A - B - C - D - E. \end{aligned}$$

In principle, we could solve these together today in class, but the work would be tedious. It's pretty easy, though, to check that the equations are satisfied when $A = 1$, $B = -1$, $C = 3$, $D = -3$, $E = 2$.

Integration

I hope that everyone in the room remembers how to calculate

$$\int \frac{A\,dx}{(x+a)^n} = A \int (x+a)^{-n}\,dx.$$

The most common situation is where $n = 1$; in that case, the integral is $A \cdot \ln|x+a| + C$.

The quadratic integrals are slightly more annoying. The things to keep in mind are:

- you need to complete the square;
- a triginometric substitution is coming your way.

On Tuesday, I wrote down the equation

$$\int \frac{-2}{x^2+x+1}\,dx = -\frac{4}{3}\sqrt{3}\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right),$$

which in fact was output to me by Sage. Now it's our job to check Sage's work.

Completing the square...

$$x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} = u^2 + a^2,$$

where $u = x + \frac{1}{2}$ and $a = \frac{\sqrt{3}}{2}$. In the integral, we substitute $u = x + \frac{1}{2}$, $x = u - \frac{1}{2}$, $dx = du$:

$$\int \frac{1}{x^2 + x + 1}\, dx = \int \frac{1}{u^2 + a^2}\, dx = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C.$$

Recall that the last equality comes from the substitution $u = a \tan\theta$, $\theta = \arctan\left(\frac{u}{a}\right)$, $du = a \sec^2\theta \ldots$.

The original integral had a factor of -2 that we shouldn't forget. Thus the integral works out to be

$$-\frac{2}{a}\arctan\left(\frac{u}{a}\right)+C=-\frac{4}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C.$$

To compare this answer with the Sage output

$$-\frac{4}{3}\sqrt{3}\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right),$$

we need to remember that $\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$. The fraction inside the arctan, for instance, becomes

$$\frac{2x+1}{\sqrt{3}}=\frac{\sqrt{3}}{3}\cdot(2x+1).$$

What happens when there's a higher power of an irreducible quadratic in the denominator of an integral? Things can get messy, but we can proceed with calm. Let's look at the simplest example $\int \frac{dx}{(1+x^2)^2}$. We would naturally make the substitution $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$. The integral then becomes

$$\int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\, d\theta.$$

The usual way to deal with the $\cos^2$ integrand is to write

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

We obtain

$$\int \frac{dx}{(1+x^2)^2} = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$
$$= \frac{\arctan x}{2} + \frac{\sin 2\theta}{4} + C$$
$$= \frac{\arctan x}{2} + \frac{\cos\theta \sin\theta}{2} + C.$$

Now

$$\cos\theta \sin\theta = \cos^2\theta \tan\theta = \tan\theta / \sec^2\theta = \frac{\tan\theta}{1+\tan^2\theta} = \frac{x}{1+x^2}.$$

Thus I believe that

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + C.$$

Can we have some validation here?

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integrate(1/(1+x^2)^2,x)
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1/2*x/(x^2 + 1) + 1/2*arctan(x)
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Looks as if we did OK.

Play hard!

Study hard!

See you after break.

Play hard!

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