Methods of Mathematics

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Linear first order ODEs

Last time we looked at first order ODEs. Today we will focus on *linear* first order ODEs. Here are some goals:

- Being able to recognize a linear first order ODE.
- Being able to solve one.
- Having some idea how such equations might arise in the life sciences. [This is homework for all of us.]

For the first point, all you need to know is the formula giving the most general linear first order ODE:

$$y' + p(t)y = q(t).$$

Here $y = y(t)$ is the unknown function of t for which we solve, while $p(t)$ and $q(t)$ are given functions of t.

Come on down to office hours! See you soon.

Special cases

If $p = 0$, then y' is given and $y = \int q(t)\,dt$ is an integral of q. For example, if $q(t) = \cos t$, we might write $y(t) = \sin t + C$.

If $q = 0$, the equation is *homogeneous*. Writing $\dfrac{dy}{dt} + p(t)y = 0$ and fooling around (doc camera or white board), we get the solution

$$y = Ce^{-\int p(t)\,dt}.$$

What's remarkable is that there's a more general formula like this in the case where q is not necessarily 0.

To motivate the “trick” that makes things work, let’s look again at the solution $y = Ce^{-\int p(t)\,dt}$ to the homogeneous problem. It’s this statement about y:

$$ye^{\int p(t)\,dt} = \text{ a constant } \iff \frac{d}{dt}\left(ye^{\int p(t)\,dt}\right) = 0.$$

Still in the homogeneous problem, we went from the ODE to be solved, namely $y' + py = 0$, to the statement that $ye^{\int p(t)\,dt}$ has derivative 0. How was this possible?

The key is to differentiate $ye^{\int p(t)\,dt}$. Note that

$$\frac{d}{dt}\left(e^{\int p(t)\,dt}\right) = e^{\int p(t)\,dt}p(t).$$

Hence

$$\frac{d}{dt}\left(ye^{\int p(t)\,dt}\right) = ye^{\int p(t)\,dt}p(t) + y'e^{\int p(t)\,dt},$$

which is

$$e^{\int p(t)\,dt}(y' + yp(t)).$$

This explains why $y' + yp(t) = 0$ if and only if $ye^{\int p(t)\,dt}$ has derivative 0.

The whole point now is that it's fruitful to multiply the equation $y' + py = q$ by an integral $e^{\int p(t)\,dt}$ even in the non-homogeneous case $q \neq 0$. Given a first order linear ODE, we fix a function $e^{\int p(t)\,dt}$ (without worrying about the usual constant of integration) and call it an *integrating factor*. The "textbook" calls this integral $I(t)$, so let's do that.

Starting with $y' + py = q$, we get $I(t)y' + I(t)py = I(t)q$; as we just saw, the LHS is the derivative of $yI(t)$. Hence the original ODE becomes

$$(yI(t))' = I(t)q,$$

which is something that we can solve quite easily:

$$y = \frac{1}{I(t)} \int I(t)q(t)\,dt.$$

Example

This example was taken from a UC Davis web page:

$$ty' + y = t^2 + 1.$$

The first thing to do is to divide by t:

$$y' + \frac{1}{t}y = t + \frac{1}{t}.$$

In our notation, $p = 1/t$ and $q = t + 1/t$. The integrating factor is

$$e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t.$$

This means that we should multiply the ODE by t; when we do so, we get back the original equation

$$ty' + y = t^2 + 1.$$

We recognize now that $ty' + y$ is the derivative of ty; we might have noticed that right away, but then many might protest that this insight was an un-motivated trick. We are able to solve the equation at this point:

$$ty = \int t^2 + 1\,dt = \frac{t^3}{3} + t + C.$$

Thus

$$y = \frac{t^2}{3} + 1 + \frac{C}{t}.$$

If we knew now that $y(1)$ were 0 (just to pick a number at random), we'd be able to find C: $C = -4/3$.

We turn now to Wikipedia for a second example:

$$y' + by = 1;$$

here, b is a non-zero constant. This example can be treated by more than one method because it's a linear DE with *constant coefficients*. Here's how we'd do it if we were not thinking about integrating factors:

Step One: Solve the corresponding homogeneous equation $y' + by = 0$. As we saw earlier ("Special cases"), the solution is $y = Ce^{-bt}$.

Step Two: Find some solution to the non-homogeneous equation. A solution comes to mind: take y to be the constant $1/b$.

Step Three: Add the two parts together. We get

$$y = Ce^{-bt} + \frac{1}{b}.$$

From the perspective of today's discussion, we solve the problem by taking $I(t) = e^{bt}$, so that the solution to the equation is

$$y = e^{-bt} \int e^{bt} \cdot 1 \, dt = e^{-bt} \left(C + \frac{1}{b} e^{bt} \right),$$

which is the same solution that we found on the previous slide.

One more example

Solve the initial value problem

$$y' - 2ty = 3t^2 e^{t^2}, \quad y(0) = 5.$$

The integrating factor is $I(t) = e^{-t^2}$. When we multiply the equation by $I(t)$, we kill off the annoying term e^{t^2} on the RHS. The general solution turns out to be $y = e^{t^2}(C + t^3)$. Plugging in $t = 0$, we find that $C = 5$.

Details on doc camera or white board.

I'll be at Crossroads Dining Commons tonight for dinner at 6:30PM. Please consider joining the group!

[Five-minute break]

Partial fractions

This is a new topic. The aim here is to be able to integrate "rational functions," which are quotients of polynomials. For example, how can we integrate

$$\frac{x^4+3x^2+1}{x^3+2x^2+2x+1} = \frac{x^4+3x^2+1}{(x^2+x+1)(x+1)}?$$

This is a curve-ball question because we will answer such questions only when the denominator has been factored into linear and quadratic factors over **R**; the quadratic factors are required to have negative discriminants (i.e., no real roots).

Step One: polynomial division

The first step is to divide the denominator into the numerator, getting a quotient and remainder. For example:

$$x^4 + 3x^2 + 1 = (x - 2)(x^3 + 2x^2 + 2x + 1) + 5x^2 + 3x + 3.$$

Then the fraction may be written as a polynomial (which we integrate) plus a rational function where the numerator has lower degree than the denominator. For example:

$$\frac{x^4 + 3x^2 + 1}{x^3 + 2x^2 + 2x + 1} = x - 2 + \frac{5x^2 + 3x + 3}{x^3 + 2x^2 + 2x + 1}.$$

Step Two: separation into partial fractions

We take the rational function that remains after division and separate it into "partial fractions" (bite-sized pieces). For example

$$\frac{5x^2+3x+3}{x^3+2x^2+2x+1} = \frac{-2}{x^2+x+1} + \frac{5}{x+1}.$$

Step Three: integration

The bite-sized pieces can be integrated by techniques from "freshman calculus," which maybe were in Math 10A? For example:

$$\int \frac{5}{x+1}\,dx = 5\ln|x+1| + C$$

and

$$\int \frac{-2}{x^2+x+1}\,dx = -\frac{4}{3}\sqrt{3}\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right).$$

The main focus is on the algebra in Step Two: how do we know that

$$\frac{5x^2+3x+3}{x^3+2x^2+2x+1} = \frac{-2}{x^2+x+1} + \frac{5}{x+1}?$$

We are supposed to know secretly that there are constants A, B and C so that

$$\frac{5x^2+3x+3}{x^3+2x^2+2x+1} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x+1}.$$

One can prove this, but we're not giving the proof here. However, if you believe that A, B and C exist, you can actually find them.

Multiply by the denominator $x^3 + 2x^2 + 2x + 1$ in the expression

$$\frac{5x^2 + 3x + 3}{x^3 + 2x^2 + 2x + 1} = \frac{Ax + b}{x^2 + x + 1} + \frac{C}{x + 1},$$

getting the polynomial identity

$$5x^2 + 3x + 3 = (Ax + B)(x + 1) + C(x^2 + x + 1).$$

We can substitute $x = -1$ to kill off the first summand; we'll get $5 = C$. Then

$$5x^2 + 3x + 3 = (Ax + B)(x + 1) + 5(x^2 + x + 1).$$

Compare coefficients of x^2; you get that $A = 0$. Plug in $x = 0$; you get that $3 = B + 5$, so $B = -2$.

Let's suppose that we wanted to decompose

$$\frac{6x^2 + 22x + 18}{(x+1)(x+2)(x+3)}$$

as a sum of partial fractions. We'd write

$$\frac{6x^2 + 22x + 18}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

and multiply to get

$$6x^2+22x+18 = A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2).$$

Set $x = -1$ to get $2 = A \cdot 1 \cdot 2$, so $A = 1$. We find similarly that $B = 2$ and $C = 3$ (I hope!).

One more example: find A, B and C if

$$\frac{9x^2 - 4x - 16}{(x^2 - x - 1)(x+3)} = \frac{Ax + B}{x^2 - x - 1} + \frac{C}{x+3}.$$

The first operation is to multiply so as to get

$$9x^2 - 4x - 16 = (Ax + B)(x + 3) + C(x^2 - x - 1).$$

Put $x = -3$ to get $77 = 11C$, so that $C = 7$. Then

$$9x^2 - 4x - 16 = (Ax + B)(x + 3) + 7(x^2 - x - 1).$$

Compare coefficients of x^2 to get $9 = A + 7$, so $A = 2$. Try $x = 0$; this gives $-16 = 3B - 7$, so $B = -3$.