Methods of Mathematics

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Last time we looked at equations like

$$a_n = a_{n-1} + 6a_{n-2};$$

the most general equation of this sort was

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_d a_{n-d}$$

where the αs are constants.

Today's perspective is that differences between successive values of a sequence are analogous to derivatives:

$$a_{n+1} - a_n = \frac{f(n+h) - f(n)}{h}$$

where $f(n) = a_n$ and $h = 1$.

Derivatives are gotten by letting $h \to 0$, and equations involving successive differences morph into equations involving derivatives.

THE f(x)

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" Mathematics is the subject where you don't know what you are talking about or whether what you say is true."
---- Bertrand Russell

THE CALCULUS OF FINITE DIFFERENCES

Richard Feynman

If we have a row of figures and subtract each number from the one following it in the list making a new list of the results of the subtraction, we will have a row of so-called " common differences". This is best illustrated by an example:

ORIGINAL NUMBERS:	1		4		10		14		13		0		6		etc
DIFFERENCES:		3		6		4		-1		-13		6			

The numbers in the difference column are found as above.
For example: 4 - 1 = 3 13 - 14 = -1

The differences of the " common differences " are called " differences of the second order". Similarly the differences of the differences of the second order are called " differences of the third order" etc. Using the same set of numbers as above for an example:

ORIGINAL NUMBERS:	1		4		10		14		13		0		6
1st DIFFERENCES:		3		6		4		-1		-13		6	
2nd DIFFERENCES:			3		-2		-5		-12		19		

There is a major analogy between ordinary (infinitesimal) calculus and the calculus of “finite differences.”

A differential equation is an equation involving an unknown function and certain of its derivatives.

- The simplest DEs involve an unknown function $y = f(x)$. These are *ordinary* differential equations. Example:

$$y'' = y' + 6y.$$

- Partial differential equations are those that involve a function of two or more variables, say $u = f(x, y)$. Example:

$$\frac{\partial u}{\partial x}(x, y) = 0.$$

- It is possible to consider *systems* of differential equations, where there are multiple unknown functions and multiple equations involving the derivatives of the functions. (This is a natural generalization.)

Cooling

Newton's law of cooling states the the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings.

Imagine a hot object (pizza?) that's placed in a relatively cool room. The temperature of the object can be denoted $T(t)$ (where t is for "time"). Suppose $t = 0$ corresponds to the moment when the object arrives in the room.

Say that T_s is the temperature of the room. It is realistic to assume that the pizza is so small relative to the room that the room's temperature is not impacted by the arrival of the pizza. In other words, the room is constantly at temperature T_s.

Newton provides us with a differential equation that's satisfied by the temperature $T(t)$.

The rate of change of the temperature of an object is $T'(t)$. Since this rate is proportional to the difference between the temperature of the object and the temperature of its surroundings, i.e., to $T(t) - T_s$, we have:

$$\boxed{T'(t) = k[T(t) - T_s] \text{ for some (negative) constant } k.}$$

After we acquire some experience in solving DEs, we'll realize that

$$T(t) = T_s + (T(0) - T_s)e^{kt}.$$

When $t = 0$, T is the initial temperature (as it's supposed to be); when $t \to \infty$, the exponential term $\to 0$ and $T \to T_s$. The buzz phrase here is "exponential cooling."

You can't figure out k without knowing some additional piece of information, such as the value of T at some time > 0.

> *At 7PM, a large pizza is taken from a* 415^oF *oven to a* 65^oF *dining room. At 7:08PM, the pizza has cooled to* 135^oF*. What is the temperature of the piece which remains at 7:16PM?*

A DE example

$$\text{Find } y = f(x) \text{ if } \frac{dy}{dx} = \sin x.$$

Here we are asked to find a function whose derivative is sin, so we write down $y = -\cos x$ or (better) $y = -\cos x + C$.

Refinement:

$$\text{Find } y \text{ if } \frac{dy}{dx} = \sin x \text{ and } y(0) = -29.$$

We use the solution $y = -\cos x + C$ and plug in $x = 0$ to find C: $-29 = -\cos 0 + C$, so $C = -28$.

An exponential example

$$\text{Find } y \text{ if } \frac{dy}{dx} = 13y.$$

If we remember that the derivative of e^{ax} is ae^{x}, then we can write down a solution $y = e^{13x}$. This is not the most general solution because we can multiply y by a constant and still have a solution. The most general solution is in fact

$$y = Ce^{13x}.$$

Refinement:

$$\text{Find } y \text{ if } \frac{dy}{dx} = 13y \text{ and } y(0) = -19.$$

We take the general solution $y = Ce^{13x}$ and plug in $x = 0$. Since $e^0 = 1$, we end up with $C = -19$, i.e.,

$$y = 19e^{13x}.$$

Differential equations have *general solutions* that involve parameters (the arbitrary constants). For an ordinary differential equation, the number of constants is the *order* of the DE: this is the order of the highest derivative.

For example, consider the DE

$$y'' = y' + 6y.$$

Because 3 and -2 are solutions to the quadratic equation

$$\lambda^2 - \lambda - 6 = 0,$$

Both e^{3x} and e^{-2x} are solutions to the DE. It follows (as we'll check on the board or with the doc camera) that

$$C_1 e^{3x} + C_2 e^{-2x}$$

is a solution for every choice of constants C_1 and C_2. This formula actually gives you the general solution.

Find y if:

$$y'' = y' + 6y, \quad y(0) = 2, \quad y'(0) = 3.$$

We write down $y = C_1 e^{3x} + C_2 e^{-2x}$ and try to find C_1 and C_2 so that

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The equations for this are

$$C_1 + C_2 = 2$$

and

$$3C_1 - 2C_2 = 3.$$

I did this quickly and got $C_1 = \frac{7}{5}$ and $C_2 = \frac{3}{5}$. Your mileage may vary.

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A typical first order ODE problem has the shape:

$$y' = \text{ some expression in } y \text{ and } t, \quad y(0) = \text{ some number.}$$

A problem like this is called an *Initial Value Problem* (IVP).

We think: y is constrained to "evolve" for $t > 0$ by the equation giving its derivative. The equation tells you how y is changing over time (derivative = rate of change); when you write down a solution you make everything explicit.

A **theorem** in the "textbook" says that every Initial Value Problem has a solution for some non-trivial interval of time that starts at 0. However, the solution can be non-unique. Also, it can blow up (go to ∞) after some period where it exists.

“Textbook” example of a blow-up

The IVP is (say)

$$\frac{dy}{dt} = y^2, \quad y(0) = 1.$$

We play fast and loose with derivatives and differentials:

- $\dfrac{dy}{y^2} = dt, \displaystyle\int \frac{dy}{y^2} = \int dt;$
- $\dfrac{-1}{y} = t + C;$
- $y = \dfrac{-1}{t + C};$
- $t = 0 \Longrightarrow C = -1;$
- $y = \dfrac{1}{1 - t}.$

The function y is perfectly happy between 0 and 1, but it blows up as $t \to 1$.

"Textbook" example of non-uniqueness

Try the same IVP with y^2 replaced by $y^{1/2}$ and with 0 as the initial value of y:

$$\frac{dy}{dt} = y^{1/2}, \quad y(0) = 0.$$

Doing the same steps as before, we'll come to $y = t^2/4$ as the solution. As the "book" points out, however, there are other solutions—for example, the function $y = 0$ (the identically 0 function).