Methods of Mathematics

Kenneth A. Ribet

UC Berkeley

Math 10B, Lecture #2
January 28, 2016

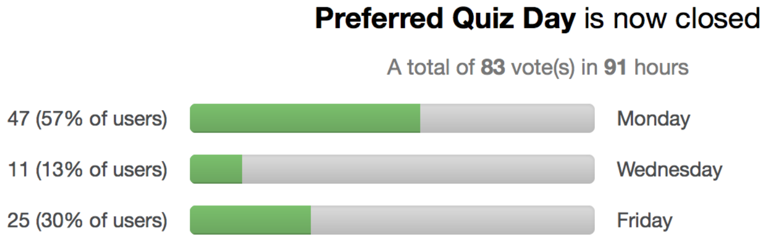
Ribet Office Hours

Monday 2:10–3:10, 885 Evans

Tuesday 10:30–noon, Student Learning Center

Thursday 10:30–11:30, 885 Evans

On Monday, February 1, I need to leave my office at 3:07 to attend a 3:10PM lecture.



Quiz tomorrow, next quiz 10 days later.

Note that only 83 people (out of 287) voted. I've gotten only 25 of your birthdays—same phenomenon?

Problems involving binomial coefficients, etc.

- How many ways are there to distribute 39 indistinguishable donuts to 287 students?
- Same question, with the proviso that no student can get more than one donut.
- How many ways are there to distribute 39 distinguishable prepaid \$20 credit cards to 287 students?
- Same question, with the previso that no student can get two or more credit cards.

- How many bit strings of length n contain exactly r 1s?
- Show that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.
- Show that $\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$.
- How many bit strings of length 12 contain three or more 1s?

Taken directly from the homework:

A box contains 14 different colored balls.

- We draw a ball at random, write down its color, and then replace it in the box. We do this 7 times in all. How many possible lists of colors are there?
- Same question, except we do not replace the balls we have drawn.

The “textbook” likes problems with balls and boxes. In subsequent slides, I’ll list a few of them.

Distinguishable balls put into distinguishable boxes

For example, the boxes might be students in the class (287 of you) and the balls might be titles (President, Vice President,…). Say there are five titles.

If students can acquire multiple titles, the number of ways to give the titles to the students is 287^5.

If student cannot have multiple titles, the number of ways is $P(287, 5)$ (as we've seen).

The "textbook" has examples of word problems where these principles apply. Read them!

Indistinguishable balls, distinguishable boxes

Let the boxes again be the students in the class. The indistinguishable balls might be math department lanyards.

If I have five lanyards and won't give two lanyards to any student, the number of ways to distribute the lanyards is $C(287, 5)$. I *choose* five lucky students.

If students can receive multiple lanyards, then we have a bagel problem. If the *i*th student gets x_i lanyards, I need to find the number of solutions in positive integers x_i to

$$x_1 + x_2 + \cdots + x_{287} = 5.$$

By Tuesday's lecture, that number is $\binom{286+5}{5} = \binom{286+5}{286}$. The lanyards are analogous to bagels and the students are analogous to the various flavors of bagel.

The power of math is its abstraction: it can view lanyards as bagels and think of the students as plain, onion, sesame, poppy,. . . .

Yesterday’s breakfast. The next scheduled events are for February 4 and February 12.

How many ways are there to solve $x+y+z+w=17$ where x, y, z and w are *positive* integers?

Let $a=x-1$, $b=y-1$, $c=z-1$, $d=w-1$. Then we need to solve $a+b+c+d=13$ in non-negative a, b, c, d. The number of solutions is $\binom{13+3}{3}$; there are 13 bagels and four kinds of bagel.

How many ways are there to solve $x + y + z + w = 17$ where x, y, z and w are *positive* integers?

Let $a = x - 1$, $b = y - 1$, $c = z - 1$, $d = w - 1$. Then we need to solve $a + b + c + d = 13$ in non-negative a, b, c, d. The number of solutions is $\binom{13+3}{3}$; there are 13 bagels and four kinds of bagel.

How many ways are there to divide a class of 287 students into k indistinguishable groups? We assume in this discussion that the groups are non-empty.

There is one way to divide the class into a single group.

There is one way to divide the class into 287 groups.

If the class has n students and there are to be k groups, the number of ways to do the division is denoted $S(n, k)$; it's a *Stirling number*.

If $k = 2$, we are trying to divide the class into two groups.
https://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind
gives a general formula

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}j^n.$$

In particular $S(n,2) = 2^{n-1} - 1$.

Why is that? Why are there $2^{n-1} - 1$ ways to divide a class of n students into two non-empty groups?

We need to assume that the class has at least two students, and we do that. Say that Alice is one of the students. We fix Alice. Then there's Alice's group and the other group. In addition to Alice, Alice's group contains at most $n - 2$ of the $n - 1$ other students.

To fill out Alice's group, we need to select a *proper* subset of the set of non-Alices. That set has $n - 1$ elements and so has 2^{n-1} subsets, including the whole set. There are $2^{n-1} - 1$ proper subsets.

As far as I can tell from the class roster, there is no student in the class whose legal name is “Alice.” However, there may be students who introduce themselves as Alice. Any takers?

Alice and Bob are traditional names in cryptography for for two people who communicate with each other.

As far as I can tell from the class roster, there is no student in the class whose legal name is “Alice.” However, there may be students who introduce themselves as Alice. Any takers?

Alice and Bob are traditional names in cryptography for for two people who communicate with each other.