Methods of Mathematics

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Math 10B, Lecture #2
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Important Announcement

The first midterm exam will be on February 16, *not* on February 18 as is indicated on the course web page and in previous announcements. (I will change the course web page.)

You spoke; I listened.

A Crossroads Lunch

Thursday, February 25 at 12:15PM.

This lunch is being organized by one of your classmates, who will post on `piazza` to try to get a count of how many people want to come.

The next breakfasts will be on January 27 (full), February 4, February 12.

A student writes:

> *I know that during class we had a vote of whether to teach by either black board or power point slides. I was wondering, is there a possibility if you can teach in both styles like how you did today. My classmates and I found it extremely helpful to have it taught in both methods.*

My response:

> *Thanks for the feedback. I'll see whether it's possible to implement your suggestion. Perhaps I'll prepare the second lecture on slides but go to the boards much more readily.*

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In a class of 287 students, there are 186 second-year students. Among the second-year students, eight have declared their majors. In the entire class, 13 students have declared their majors.

How many students in the class are either ____________ or ____________?

For a specific example: how many students are either (a) undeclared or (b) second-year students?

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This question involves two concepts (and therefore should be regarded as pedagogically suspect). The first concept is that of a *complement*.

Let S be the set of students in the class (our class, actually) and let A be the subset of S consisting of students who have declared their majors.

The *complement* of A (in S) is the set A^c consisting of elements of S that are *not* in A. One writes sometimes

$$A^c = S \setminus A \text{ or } A^c = S - A.$$

The number of undeclared students in the class is $287 - 13 = 274$. In general:

$$|A^c| = |S| - |A|,$$

where $|\cdot|$ is used to denote the number of elements in a set.

The main thing that we need to do is to relate the number of elements of the union of two sets to the number of elements of the individual sets and the number of elements of the intersection. The formula is:

$$|B \cup C| = |B| + |C| - |B \cap C|.$$

This is called "inclusion-exclusion" because you count up the elements in B and in C and then "exclude" (subtract off) the elements in the intersection because you would otherwise be counting them twice.

In the application, $B = A^{\mathrm{c}}$ might be the set of undeclared students and C might then be the set of second-year students.

We have $|B| = 274$, $C = 186$. What is $B \cap C$? This is the set of undeclared second-year students:

$$B \cap C = C - C \cap A = C \setminus A.$$

Since C has 186 elements and $C \cap A$ has eight elements, $|B \cap C| = 186 - 8 = 178$.

Thus

$$|B \cup C| = |B| + |C| - |B \cap C| = 274 + 186 - 178 = 274 + 8 = 282.$$

Of course we can see this as follows: we count the students who are either undeclared or second-year students by adding the number of undeclared students (274) to the number of declared second-year students (8).

What if there are three sets instead of two? Suppose there are four sets? n sets?

For three sets A, B and C, we can talk our way through the formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The elements in the triple intersection are added in three times and then subtracted off three times. They have to be added in at the end so that they count exactly once.

How many integers between 1 and 100 are divisible by at least one of the numbers 2, 3, 5?

Let A, B and C be the sets of integers in the range $(1 \cdots 100)$ that are divisible by 2, 3 and 5, respectively. Then $|A| = 50$, $B = 33$, etc., etc. The set $A \cap B \cap C$ consists of multiples of 30 in the relevant range, so $|A \cap B \cap C| = 3$.

My answer to the question is 74. Is that right?

Here's another situation where the triple intersection comes in. (I intend to explain this mainly on the boards.)

Suppose three students come up and hand me their iPhones. I then want to return phones to students, but not necessarily to their rightful owners. The map sending phones to people can be modeled by a function

$$f : \{ 1, 2, 3 \} \longrightarrow \{ 1, 2, 3 \};$$

$f(i) = j$ means that the ith phone (the one originally belonging to student i is given to student j.

The number of such functions is $3 \times 3 \times 3$ because there are three possible recipients for each phone.

We are interested only in functions that are *one-to-one* and *onto*. This means that no student gets two or more phones and that every student gets at least one phone. (The two conditions are equivalent because the number of students is equal to the number of phones.)

The functions like this are called *permutations* of $\{1, 2, 3\}$.

The number of permutations of $\{1, 2, 3\}$ is $3! = 6$.

The number of permutations of $\{1, 2, 3, \ldots, n\}$ is $n!$

Among the six permutations of $\{1, 2, 3\}$, how many that the property that $f(i) \neq i$ for $i = 1, 2, 3$? This condition corresponds to the requirement that no student gets his or her own phone back.

We can calculate this by hand: $f(1)$ can be either 2 or 3. To fix ideas, say $f(1) = 2$. Then $f(2)$ can't be 1 because $f(3)$ would be forced to be 3. Thus $f(2) = 3$ and $f(3) = 1$. Similarly, if $f(1) = 3$, then $f(2) = 1$ and $f(3) = 2$. Thus there are two such functions.

The fraction of permutations of $\{1, 2, 3\}$ that have no "fixed points" is $2/6 = 1/3$. It is an interesting problem to calculate this fraction when 3 is replaced by n. (See `https://en.wikipedia.org/wiki/Derangement`.)

We can re-do the calculation by inclusion–exclusion:

Let A be the set of permutations f that take 1 to 1, i.e., are such that $f(1) = 1$. Let B be the set of permutations that take 2 to 2; let C be the set of permutations that take 3 to 3. Then $A \cup B \cup C$ is the set of permutations with at least one fixed point. If $|A \cup B \cup C|$ can be computed to be 4, then there are $6 - 4 = 2$ permutations with no fixed point. In the formula

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|,$$

the double intersections and triple intersections consist only of the identity function. Hence the right-hand side is

$$2+2+2-1-1-1+1 = 6-3+1 = 4,$$

as expected.

For two sets, we add and then subtract. For three sets, we add, subtract and add. For four, we add, subtract, add and subtract. You can imagine the general picture. You can see it on slide 32 of the `Combinatorics` textbook.

Pigeonhole Principle

There are lots of ways to say this. First of all, pigeonholes are mailbox slots. Imagine placing a stack of letters into pigeonholes. The principle is: *If there are more letters than pigeonholes, at least one pigeonhole has to get two or more letters.*

Mathematically, if we have a function $f : S \to T$, where S and T are finite sets, and if $|S| > |T|$, then there are two different elements of S with the same image in T. In other words, there are $s, s' \in S$, $s \neq s'$, such that $f(s) = f(s')$.

Another way to state the conclusion: There is an element t of T for which the equation $f(x) = t$ has two or more solutions in S.

Yet another formulation: if $f : S \to T$ is a function between finite sets that consistently takes different elements of S to different elements of T, then $|T|$ is at least as big as $|S|$.

Here's a concrete conclusion: In this class, there must be two different students who student ID numbers end with the same three digits. Here S is the set of students in the class T is the set of integer strings from 000 to 999, and $f : S \to T$ is the function that takes s to the last three digits of s's SID.

In fact, two SIDs in the class are 25706010 and 25984010.

This was a major screwup on my part because there are 1000 such strings but only 287 students. The pigeonhole principle does *not* apply. The situation is instead like that for birthdays. (See the next slide.)

Since there are fewer than 366 students in the class, the Pigeonhole Principle does *not* guarantee that there are two students with the same birthday. However, you probably have heard of the Birthday Paradox: it's actually extremely likely that two of you have the same birthday. Since you are all roughly the same age, it's even highly likely that two of you were born on the very same day.

You can all email me your birthdays. I'll collate and write to groups of you who have the same birthday. Few of you have done this. So far there has been no collision!

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Rosen: Show that for every integer n there is a positive multiple of n that has only 0s and 1s in its decimal expansion.

Let n be a positive integer. Consider the $n+1$ integers 1, 11, 111,...,$111\cdots1$, where the last integer in the series has $n+1$ digits. Imagine dividing each integer in this series by n: we get a quotient and a remainder. The remainder is an integer between 0 and $n-1$, so there are only n possible remainders.

Pigeonhole: two of the integers in the series must have the same remainder on division by n. As a result, the difference of those two integers will have remainder 0 on division by n. In other words, the difference will be a multiple of n. If we make the difference by subtracting the smaller of the two numbers from the larger, we get a positive integer that's a multiple of n. The difference will have only 1's and 0's in its decimal expansion.

A numerical example

Suppose that $n = 7$. Then 1111110 (six 1's followed by one 0) is a multiple of 7; in fact it's 7×158730.

Generalization: If p is a prime number different from 2 and 5, the p-digit number $11 \cdots 110$ is a multiple of p. To see this, you need something called Fermat's Little Theorem, which you'll find in most discrete math textbooks.

Several of you have asked for more explanation as to how the pigeonhole principle is being applied in this situation.

First of all, the key principle is that a function $f : S \to T$ between two finite sets is not 1-1 if S has more elements than T. Not being 1-1 means that there are two different elements of S that have the same image in T. Using more symbols, we can say: there are $s, s' \in S$ with $s \neq s'$ and $f(s) = f(s')$.

In our situation, S is the set of numbers 1, 11, ..., $11 \cdots 1$; there are $n + 1$ of these numbers. T is the set consisting of 0, 1, 2, 3,... $n - 1$. Thus S has more elements than T, which has n elements. The function f takes an integer $1 \cdots 1$ to its remainder on division by n.

A homework problem

"Let S be a set of six positive integers whose maximum is at most 14. Show that the sums of the elements in all the nonempty subsets of S cannot all be distinct."
There are 6 elements of S, so S has $2^6 = 64$ subsets, including the empty set. (We write $|2^S| = 64$.) Thus there are 63 non-empty subsets.

To each non-empty subset T of S, we associate the sum of the elements of T. The sum is at least 1 and at most $9 + 10 + 11 + 12 + 13 + 14 = 69$. Thus the sums lie in a set with 69 elements. Since $69 \geq 63$, we have no way to use the Pigeonhole Principle.

There is an amazing trick, however. We ignore the set S itself and consider only non-empty *proper* subsets T of S. There are 62 of these subsets. Since T has at most 5 elements, the sum of the elements of T is at most $10 + 11 + 12 + 13 + 14 = 60$. Because $60 < 62$, we now win. . . . This is quite subtle.

Maybe not: the entire set S has a sum that cannot be duplicated by a proper subset. In the same vein, the empty set has a sum (0, by definition) that cannot be duplicated by a non-empty subset. Thus it makes 100% sense to remove these two extreme sets from consideration.

The wording of the problem forces us to ignore the empty set, but we should realize quickly that the full set S will analogously be of no use to us.