Methods of Mathematics

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UC Berkeley

Math 10B, Lecture #2
January 19, 2016

Welcome!

My `https://math.berkeley.edu` web page for the course is at `https://math.berkeley.edu/~ribet/10B/`

The bCourse page for the class:
`https://bcourses.berkeley.edu/courses/1408209`

Note that the midterm exams are on February 18 and March 31 during class time.

Come to our breakfasts!

This is a photo of the last breakfast of the fall semester.

Upcoming breakfasts: Jan. 27 (full!), Feb. 4 and Feb. 12.

This is a photo of the group at this morning's breakfast.

Today's discussion concerns the basic principles of counting. Roughly speaking, we need to discuss pp. 6–25 of the 124-page "texbook" Combinatorics.pdf.

We could do this by racing through the twenty different slides. Bad idea! We'll do what we can do. Inevitably, however: *some of the material will not be discussed at all today in class*. Thus we will all need to study the "textbook" offline.

Optimally, we will also begin poring over the material on the inclusion–exclusion principle and on the pigeonhole principle. This material is presented on slides 26–46 and will be discussed in class on Thursday.

The best way to do well in this class is to read the material ahead of time. Above all, do not think that it's OK to fall behind because you can catch up right before quizzes and exams. That would be a major disaster (IMHO).

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By 3:30 today, we want to be stars at answering questions like these:

- How many license plates can we make if license plates have three numerical digits (0–9) followed by three letters of the alphabet (A–Z)? [Example: `454MGI`]
- How many license plates can we make if plates have seven characters, each one of which can be a digit or a letter? [Example: `G37HZ1A`]
- How many palindromic bit strings are there of length 22? of length 21? [`1101000111001110001011`]
- This class has 287 enrolled students (approximately). How many ways are there to choose a subset of the class?
- This class has 287 enrolled students, 14 waitlisted students, 5 GSIs, 1 instructor (me!) and 2 SLC instructors (Mike and Assal). How many ways are there to choose a member of the Math 10B, Lecture 2 community? [it's the sum]

More questions

- You need to choose a password, which is 6 to 8 characters long, where each character is a lowercase letter or a digit. How many possible passwords are there?
- How many five-letter words (strings of lower-case letters) contain exactly one “e”?
- A playoff between two teams ends when one team has won three games. In how many different ways can the playoff unfold? [The answer is 20.]
- Same as the previous question, with “three” replaced by N. [I don’t know the answer offhand.] The case $N = 4$ is pretty familiar (World Series, etc.).

How many license plates can we make if license plates have three numerical digits (0–9) followed by three letters of the alphabet (A–Z)?

There are 10 choices for the first character, 10 for the second,. . . , 26 for the sixth.

The answer is then "obviously"

$$10 \times 10 \times 10 \times 26 \times 26 \times 26 = \text{whatever}.$$

Actually whatever $= 17576000$.

This is the *multiplication rule* in action.

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How many license plates can we make if plates have seven characters, each one of which can be a digit or a letter?

There are 36 choices for each character. By the multiplication rule, the answer is 36^7.

How many palindromic bit strings are there of length 22? of length 21?

For problems like this, it is simplest to consider analogous smaller problems. How many palindromic strings are there of length four, for example?

Such a string looks schematically like ABBA, where A and B are binary digits (0 or 1). There are two choices for A and two for B, so there are $2 \times 2 = 4$ possibilities.

Palindromic strings of length five look like ABCBA, so there are $2 \times 2 \times 2 = 8$ of them. There are also eight palindromic strings of length 6; those have the shape ABCCBA.

So we can do this now for strings of length $2n$ and length $2n + 1$, right?

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This class has 287 enrolled students. How many ways are there to choose a subset of the class?

To choose a subset of the class, we can imagine the students listed alphabetically (from ABxxx to ZOxxx). To make up a subset of the class, we can go down the list of students, saying “not in the set” or “in the set” for each student.

For each student, there are two choices. The number of sets is $2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 = 2^{287}$.

Note that a subset of the class is the same thing as a binary string of length 287: as we go down the list of students, we write “0” if the student is not in the set and “1” if the student is in the set.

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How many of you like it when I lecture from slides like this?

How many of you prefer that I write on the board?

Feedback: send me email, post on piazza, tell me in office hours,. . . .

This class has 287 enrolled students, 14 waitlisted students, 5 GSIs, 1 instructor (me!) and 2 SLC instructors. How many ways are there to choose a member of the Math 10B, Lecture 2 community?

The community has $287 + 14 + 5 + 1 + 2 = 309$ members. It's not rocket science: there are 309 ways to choose one element from a set of size 309.

How many license plates can we make that are *either* three numerical digits (0–9) followed by three letters of the alphabet (A–Z) *or* seven characters, each one of which can be a digit or a letter?

The answer is the *sum* of the answers to the first two problems. You are seeing the **sum rule** in action.

How many ways are there to choose two members from our Math 10B community?

We can think of choosing *first* and *second* member of the community. There are 309 ways to choose the first member and then 308 ways to choose the second member. It is tempting to think that the answer is 309×308.

But this is not quite correct. Given a set of two members, we have two ways to "order" the set, i.e., to choose a first member and a second member. If we choose Wendy first and Bob second, we get the same set that we would get by choosing Bob and then Wendy.

Thus the correct answer is $(309 \times 308)/2$. The general topic here: binomial coefficients (to be discussed on January 28).

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You need to choose a password, which is 6 to 8 characters long, where each character is a lowercase letter or a digit. How many possible passwords are there?

We add up the numbers corresponding to 6-, 7- and 8-character passwords:

$$36^6 + 36^7 + 36^8.$$

We just use the sum rule; there's nothing more subtle going on.

How many five-letter words (strings of lower-case letters) contain exactly one "e"?

This is intended to be another application of the sum rule.

How many 5-letter words have an "e" in the first place and no further e's? Answer: $25 \times 25 \times 25 \times 25 = 25^4$.

How many 5-letter words have an "e" in the second place and no further e's? Answer: the same as before.

$$\cdots$$

The answer is thus $25^4 + 25^4 + \cdots + 25^4 = 5 \cdot 25^4$.

A playoff between two teams ends when one team has won three games. In how many different ways can the playoff unfold?

Say that the teams are W (Warriors) and L (Lakers). We first count the number of situations in which the Warriors win:

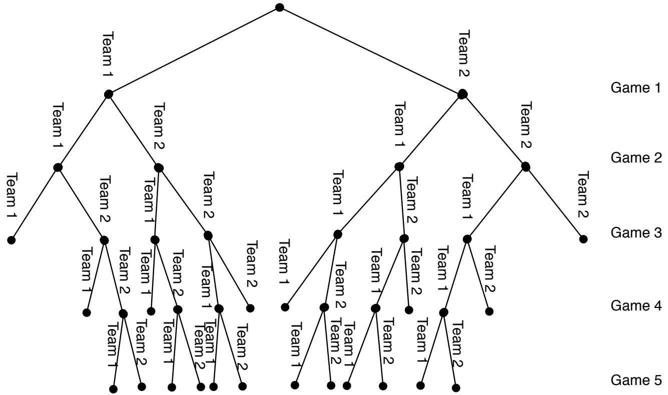
There is one way in which the Warriors win in three (straight) games; we can write it symbolically as WWW.

If the Warriors win in 4 games, the outcome string looks like xxxW, where exactly one of the x's is an L. Here there are three choices for the placement of the L.

If the Warriors win in 5 games, the outcome string looks like xxxxW, where two of the four x's are L's. The number of ways to choose two things out of four is $(4 \cdot 3)/2 = 6$.

The number of ways in which the Warriors can win is thus $1 + 3 + 6 = 10$. The number of ways in which the series can unfold is then $2 \times 10 = 20$.

In the notes, the authors propose to do this problem by enumerating the possible outcomes:



I need to count with my finger to see that there are 20 ends to the story. How do I know I haven't missed one of the ends or miscounted in some other way??