Methods of Mathematics

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Math 10B
February 4, 2016

We are moving!

Our new classroom is F295 Haas.

The effective date of the move is next Tuesday, February 9.

Ribet Office Hours

Monday 2:10–3:10, 885 Evans

Tuesday 10:30–noon, Student Learning Center

Thursday 10:30–11:30, 885 Evans

Upcoming breakfast

8AM, Monday, February 22: I annouced Wednesday, February 24, but the Club is very busy that day. February 22 seems like a good alternative.

This will be at the Faculty Club.

I announced the “event” only Tuesday evening; there are still a half-dozen places.

Send me email if you’d like to come.

Lunches

Monday, February 8, 12:30PM at the Faculty Club. Let me know if you'd like to come.

Tuesday, February 9: We could go to a DC after my SLC office hour. We'd need a volunteer organizer.

This morning's breakfast

We had 22 diners at breakfast this morning. Here's a view of *some* of them. Show up at the Faculty Club and you meet all sorts of interesting people.

This morning's breakfast

Our ceremonial photo toward the end of breakfast.

Probability

Today's topics are *Probability* and *Conditional probability*. The "textbook" now changes to the second of the .pdf's on bCourses; this is the section on Discrete Probability Theory.

The first object of study is a *probability space* Ω. This is a "discrete" set; usually, it's a finite set. It comes equipped with a probability function

$$P : A \mapsto P(A), \qquad 0 \leq P(A) \leq 1,$$

where A runs over the subsets of Ω. There are some axioms:

- $P(\Omega) = 1$;
- $P(\emptyset) = 0$;
- $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.

The essential case is that where Ω is finite. Then we can write $P(x)$ for $P(\{x\})$ for $x \in \Omega$. The last axiom then tells us that

$$P(A) = \sum_{x \in A} P(x)$$

for each subset A of Ω.

Terminology: Subsets of Ω are called "events." For $A \subseteq \Omega$, we say that $P(A)$ is the probability that A "occurs."

If A^c is the complement of A, then Ω is the disjoint union of A and A^c, so $1 = P(\Omega) = P(A) + P(A^c)$. Hence

$$P(A^c) = 1 - P(A).$$

Uniform distribution

For most of the initial examples, we will assume that

$$P(x) = \frac{1}{|\Omega|} \text{ for all } x \in \Omega.$$

Thus $P(A) = |A|/|\Omega|$ for $A \subseteq \Omega$. See https://en.wikipedia.org/wiki/Uniform_distribution_(discrete) for some discussion if you are so inclined.

Example: If we toss a random US coin, we expect that the coin lands on both Heads and on Tails with probability 1/2. We might model this situation with the space

$$\Omega = \{\,\mathrm{T}, \mathrm{H}\,\}$$

with the uniform distribution

$$P(\mathrm{H}) = \frac{1}{2} = P(\mathrm{T}).$$

A coin like this is referred to as a *fair* coin.

A coin where T and H are different from 1/2 is called *biased*.

To see how discrete probability is used in practice, it is best to do examples like those in the "text."

What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 5?"

The dice have six sides, with "numbers" 1–6. (I assume you've all seen dice.) It is traditional to distinguish between the two dice (left-hand and right-hand dice, say) and to imagine that they are both *fair*. We model this with the uniform space

$$\Omega := \{ (i,j) \mid 1 \le i, j \le 6 \}.$$

We compute

$$P(\{ (1,4), (2,3), (3,2), (4,1) \}) = \frac{4}{36} = \frac{1}{9}.$$

What is the probability of getting a flush in a hand of poker?

The answer is $\frac{N}{D}$, where D is the total number of possible poker hands and N is the number of such hands that represent a *flush*. This turns out to be about 0.0019654.

We have $D = \binom{52}{5} = 2598960$ (a 7-digit number).

A *flush* is a hand where all five cards are of the same suit (e.g., hearts) but the cards are *not* in sequence. If the cards are in sequence (e.g., the 3, 4, 5, 6 and 7 of diamonds), then we're dealing with a "straight flush" or a "royal flush." These possibilities are excluded by the simple term "flush."

Perhaps you learned something in school today.

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However, look at Problem 15 in the homework due on February 10: “What is the probability that a five-card poker hand is a flush (all cards are the same suit)?"

In that problem, it is apparent that the term “flush” is to be interpreted broadly so as to include straight flushes and royal flushes.

After you read the next slide or two, you’ll know how to do the problem!

To make a poker hand where all cards are of the same suit, we choose a suit—four possibilities for that—and then choose five of the 13 cards of that suit. Thus there are

$$4 \cdot \binom{13}{5}$$

hands that are “flushes or above.”

In a given suit, how many of these “flushes+” are in fact straight or royal? The special flushes of a given suit are determined by their low cards, which can be ace, 2, 3,…,10; there are ten possibilities. Thus I think that the number of flushes is

$$4(\binom{13}{5} - 10) = 5108.$$

I was delighted when I saw that my number agrees with
http://people.math.sfu.ca/~alspach/comp18/ and
https://math.berkeley.edu/~ribet/55/poker.pdf.

There is no real substitute for positive feedback.

Conditional probability

For $A, B \subseteq \Omega$, the probability of "A given B" is defined to be

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

when $P(B)$ is non-zero. The idea is that "given B" forces us to work inside B and thus to replace A by its intersection with B. By forming the fraction, we are figuring out what part of A is inside B.

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Answer: $\dfrac{\binom{5}{3}}{2^5}$.

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A fair coin is flipped five times. What is the probability of getting three heads, given that the number of heads is odd?

Let A_i be the event that there are exactly i heads. Then $A = A_3$ and $B = A_1 \cup A_3 \cup A_5$. We have $A \subset B$, so $A \cap B = A$ in this case. We find that $P(A|B)$ is

$$\frac{P(A_3)}{P(A_1) + P(A_3) + P(A_5)} = \frac{\binom{5}{3}}{\binom{5}{1} + \binom{5}{3} + \binom{5}{5}} = \frac{10}{5 + 10 + 1}.$$

What is the probability of having a flush in a hand of poker, given that we do not have three of a kind?

If A is the event of getting a flush and B is the event of having three of a kind, then $A \cap B = \emptyset$, which is to say that $A \subseteq B^c$. The desired condtional probability is

$$\frac{P(A)}{P(B^c)} = \frac{P(A)}{1 - P(B)}.$$

We computed $P(A)$ in the earlier discussion. We can compute $P(B)$ in a similar way.

In class, we computed the number of 5-card hands that are three of a kind. Here was my reasoning:

First we choose the “rank” (or kind of card) that we have three of; there are 13 possibilities. Then we choose three out of the four cards of this rank—4 possibilities. Then we have to choose the two “other” cards. My way of doing that was to think of the ways of choosing two cards from the 48 cards that aren’t of the given rank and then subtract off the ways of choosing two cards that are of the same rank as each other. (We don’t want to have a full house.) The expression that I left on the board was

$$13 \cdot 4 \left(\binom{48}{2} - 12 \binom{4}{2} \right).$$

On the web, I found the number 54972 and asked whether my expression equals 54972.

It doesn't: my expression equals 54912. However, if you look at https://math.berkeley.edu/~ribet/55/poker.pdf, the author (my former GSI Loren Looger) writes $\binom{13}{3} \cdot 3 \cdot \binom{4}{3} \cdot 4^2$ for the number of three-of-a-kind hands and then computes this product to be 54972. It isn't: it's also 54912. I think that Loren made a transcription error. Actually, I noticed this same error recently when teaching Math 55.