Methods of Mathematics

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UC Berkeley

Math 10B
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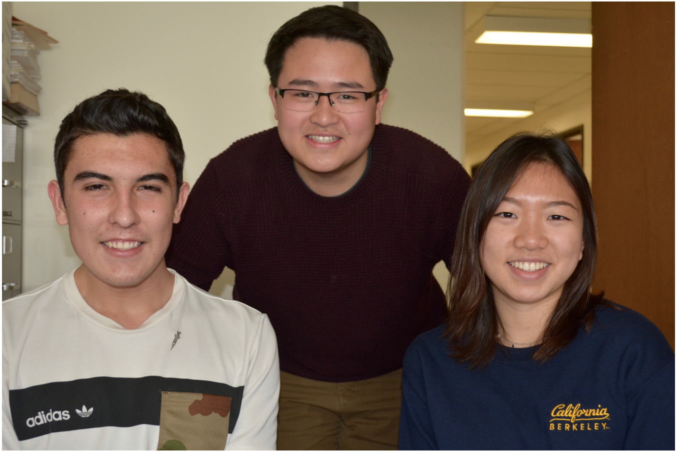
Monday 2:10–3:10 and Thursday 10:30–11:30 in Evans



Tuesday 10:30–noon at the SLC

Breakfasts

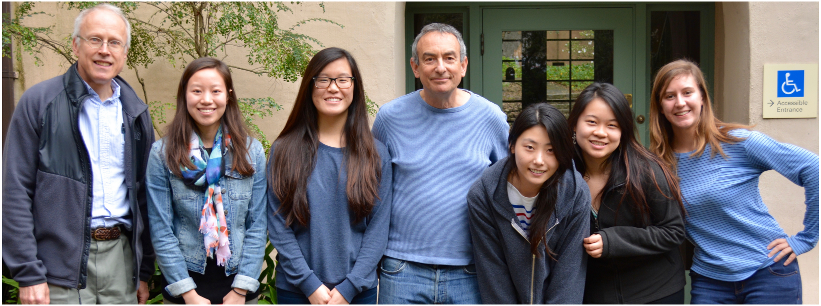
The March 3 breakfast (8AM) is close to full. After March 3, the next breakfast will be on March 18, the last day of classes before spring break. The time will again be 8AM. Send email to sign up.

Crossroads lunch

We had a Crossroads lunch today.

Another view of the same activity.

Our next pop-up Faculty Club lunch will be at 12:30PM tomorrow. It is marginally helpful if you let me know in advance that you plan to come. *However, you can just show up!*



All are welcome.

Expected value

If $X : \Omega \to \mathbf{R}$ is a random variable, the *expected value* of X is the average value of X:

$$E[X] := \sum_{\omega \in \Omega} X(\omega) P(\omega) = \sum_{x \in \mathbf{R}} x \cdot P(X = x).$$

You pass from the first sum to the second sum by grouping together all ωs for which $X(\omega)$ has a given value, x.

Because $E[X] = \sum_x x \cdot P(X = x)$, we can compute the expected value of a random variable just by knowing its probability distribution.

Example: Poisson and geometric distributions

As we saw at the end of class on Tuesday, if

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \text{ for all } k \geq 0,$$

then $E[X] = \lambda$.

For the geometric distribution,

$$P(X = k) = q^k p \text{ for all } k \geq 0,$$

where $q = 1 - p$. Then

$$E[X] = \sum_{k=0}^{\infty} kq^k p = p \sum_{k=0}^{\infty} kq^k.$$

To evaluate the sum, we differentiate the formula

$$1 + t + t^2 + \cdots = \frac{1}{1-t}$$

with respect to t and then multiply by t, thereby getting

$$t + 2t^2 + 3t^3 + 4t^4 + \cdots = \frac{t}{(1-t)^2}.$$

Then

$$p \sum_{k=0}^{\infty} kq^k = \frac{pq}{p^2} = \frac{1-p}{p}.$$

As an exercise, try to calculate the expected values of the other "standard" distributions that we discussed on Tuesday.

Properties of expected value

- $E[cX] = cE[X]$ when c is a constant;
- $E[X_1 + X_2] = E[X_1] + E[X_2]$ when X_1 and X_2 are random variables;
- $E[XY] = E[X]E[Y]$ when X and Y are *independent*.

Only the third property is subtle. We'll do an example to see how its proof plays out.

Two spinners

As in one of the HW problems, suppose we have two independent spinners: spinner #1 has values 1, 2, 3, while spinner #2 has values 1, 2, 3, 4. We do *not* assume that the spinners give their possible numbers with equal probability. The sample space is the set of pairs (i, j) with $1 \le i \le 3$, $1 \le j \le 4$; it has 12 elements. The two independent random variables are $X : (i, j) \mapsto i$ and $Y : (i, j) \mapsto j$. The expected values $E[X]$ and $E[Y]$ are *unknown* because we allow the spinners to be biased.

Nonetheless, we want to see why $E[XY] = E[X]E[Y]$.

For $i = 1, 2, 3$, let p_i be the probability that the first spinner lands on i; for $j = 1, 2, 3, 4$, let q_j be the probability that the second spinner lands on j. Then

$$E[X] = p_1 + 2 \cdot p_2 + 3 \cdot p_3, \quad E[Y] = q_1 + 2 \cdot q_2 + 3 \cdot q_3 + 4 \cdot q_4,$$

so that

$$\begin{aligned} E[X]E[Y] = 1 \cdot p_1 q_1 &+ 2 \cdot (p_2 q_1 + p_1 q_2) + 3 \cdot (p_3 q_1 + p_1 q_3) \\ &+4 \cdot (p_2 q_2 + p_1 q_4) + 6 \cdot (p_3 q_2 + p_2 q_3) \\ &+8 \cdot p_2 q_4 + 9 \cdot p_3 q_3 + 12 \cdot p_3 q_4. \end{aligned}$$

Recall from HW that the possible values of XY are 1, 2, 3, 4, 6, 8, 9, 12. Thus

$$E[XY] = \sum_k k \cdot P(XY = k),$$

where k ranges over the 8 possible values of XY.

We want to check, for example, that $P(XY = 4)$ is the sum $p_2 q_2 + p_1 q_4$. The event "$XY = 4$" is the disjoint union of these events:

- A: "$X = 2$ and $Y = 2$";
- B: "$X = 1$ and $Y = 4$".

Accordingly,

$$P(XY = 4) = P(A) + P(B).$$

By the definition of independence of random variables, each of the probabilities on the right side of the equation is a product. For example,

$$P(X = 2 \text{ and } Y = 2) = P(X = 2)P(Y = 2).$$

In our notation $P(X = 2) = p_2$ and $P(Y = 2) = q_2$. Therefore $P(A) = p_2 q_2$. Similarly, $P(B) = p_1 q_4$. Thus $P(XY = 4)$ is indeed the required sum.

Covariance

If X and Y are not necessarily independent, we define

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y]$$

as a measure of non-independence. Independent random variables have covariance = 0, but *not conversely*.

`stackexchange`: Suppose X has values ± 1 and $P(X = 1) = P(X = -1) = \frac{1}{2}$; suppose that $Y = 0$ when $X = -1$ but Y is ± 1 with equal probabilites if $X = +1$. Then $E[X] = E[Y] = E[XY] = 0$ but X and Y are not independent.

The details

Let $\Omega = \{a, b, c\}$. Define $P(a) = 1/2$, $P(b) = P(c) = 1/4$. Define $X(a) = -1$, $X(b) = X(c) = +1$ and $Y(a) = 0$, $Y(b) = -1$, $Y(c) = +1$. One checks (do it!) that $E[X] = E[Y] = 0$. Also, $XY = Y$, so $E[XY] = 0$ and thus $\mathrm{Cov}[X, Y] = 0$.

However, $P(X = 1 \text{ and } Y = -1) = 1/4$ while $P(Y = -1) = 1/4$ and $P(X = 1) = 1/2$. Thus X and Y are not independent because $1/4$ is not the product of $1/4$ and $1/2$.

A common notation for $E[X]$ is μ:

$$\mu = E[X].$$

That's because μ is the Greek **m**, and "m" stands for "mean."

Variance

The variance of a random variable X is defined by

$$\mathrm{Var}[X] = E[(X-\mu)^2],$$

where μ is the expected value of X. The variance of X is the average of the square of the deviations of X from its average value. A famous formula is

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2.$$

To prove it, we expand $(X-\mu)^2$ and use the linearity of expected value:

$$\begin{aligned} E[(X-\mu)^2] &= E[X^2] - 2\mu E[X] - E[\mu^2] = E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - E[X]^2. \end{aligned}$$

Variance of a sum

If X and Y are random variables, then

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y].$$

This follows from the "famous formula" on the previous slide, the definition of covariance, and the formula for the square of a sum.

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Then the variance of a sum of two *independent* random variables is the sum of the variances of the two variables.

Variance of Poisson distribution

If the values of X are distributed according to the Poisson rule with paramter λ, then

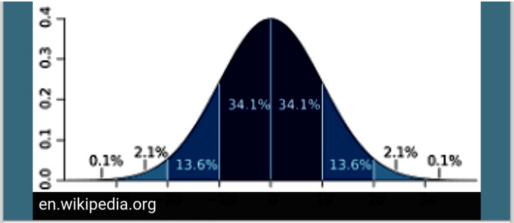
$$E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} (j+1) \frac{\lambda^j}{j!},$$

The sum on the right is the sum of two terms, one of which is e^λ. The other is λe^λ, as we saw on Tuesday. Hence $E[X^2] = \lambda(1+\lambda)$, and then

$$\mathrm{Var}[X] = \lambda(1+\lambda) - \lambda^2 = \lambda$$

because $E[X] = \lambda$.

Standard Deviation



"The **Standard Deviation** is a measure of how spread out numbers are. Its symbol is σ (the greek letter sigma) The formula is easy: it is the square root of the Variance."

Standard Deviation and Variance - Math is Fun