Methods of Mathematics

Kenneth A. Ribet

UC Berkeley

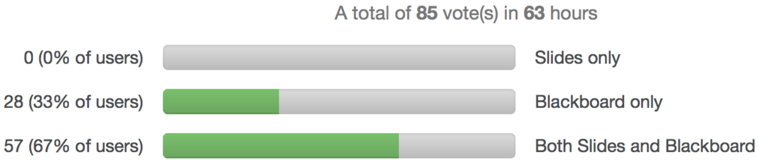
Math 10B
February 2, 2016

Ribet Office Hours

Monday 2:10–3:10, 885 Evans

Tuesday 10:30–noon, Student Learning Center

Thursday 10:30–11:30, 885 Evans



The aim is to have the slides guide the discussion and for most of the math work to be done on the chalkboards.

Also, problems that appear on the slides will be followed by **long pauses.**

Mathematical induction

Several students have asked me about mathematical induction (usually just called “induction” for short). Here is a summary:

The positive integers are the numbers 1, 2, 3,. . . .

In many countries the *natural numbers* are the numbers 0, 1, 2,. . . .

In North America, some people think that the natural numbers are the same thing as the positive integers. Others prefer to have the natural numbers begin with 0. I like to include 0 and will try to remind you of that whenever it might be an issue.

Well ordering

Mathematical induction depends on the following fact (known as the well-ordering principle):

Every non-empty set of positive integers has a smallest element.

See `https://en.wikipedia.org/wiki/Well-ordering_principle` for some discussion.

In induction, you have a proposition of mathematics that has a "paramter" n in it, for example the statement:

The sum of the first n positive integers is $\frac{n(n+1)}{2}$.

The proposition can be called $P(n)$. The principle of induction is that to prove $P(n)$ for all n, you need prove only:

- $P(1)$ (the "base case")
- $P(k) \longrightarrow P(k+1)$ for all $k \geq 1$.

Heuristically, if you know this, you can use the implication to deduce $P(2)$ from $P(1)$, then use the implication to deduce $P(3)$ from $P(2)$, then use it again to deduce $P(4)$, and so on and so forth. "We continue in this matter."

Induction nails down the logic that justifies this reasoning.

Why induction works

Assume that we have established $P(1)$ and that $P(k)$ implies $P(k+1)$ for all k. Then the claim is that $P(n)$ is true for all n.

To see this, we consider the set S of positive integers n for which $P(n)$ is false. If S is empty, $P(n)$ is true for all n. If not, S has a least element—call it ℓ. Then $P(\ell)$ is false but $P(n)$ is true for all positive integers n less than ℓ. Since $P(1)$ is true, $\ell \neq 1$. Thus $\ell - 1$ is a positive integer. Because ℓ is the smallest integer for which the proposition is false, $P(\ell - 1)$ is true. Taking $k = \ell - 1$ and using $P(k) \to P(k+1)$, we see now that $P(\ell)$ is true. This is a contradiction, so it must be the case that S is empty, i.e., that $P(n)$ is true for all n.

To prove $P(k) \longrightarrow P(k+1)$, one assumes $P(k)$ and shows that one can deduce $P(k+1)$. Thus in induction proofs, one often writes:

"Assume $P(k)\ldots$."

This is mega-confusing to the unititiated: why is one assuming what one wants to prove?!

Fortunately, we are the initiated.

To prove $P(k) \longrightarrow P(k+1)$, one assumes $P(k)$ and shows that one can deduce $P(k+1)$. Thus in induction proofs, one often writes:

"Assume $P(k)\ldots$."

This is mega-confusing to the unititiated: why is one assuming what one wants to prove?!

Fortunately, we are the initiated.

A first example

Consider:

$$P(n)\text{: the sum of the first } n \text{ positive integers is } \frac{n(n+1)}{2}.$$

The statement $P(1)$ is true because the sum of the first one positive integer(s) is 1, which agrees with $\frac{1\cdot 2}{2}$.

To prove $P(k) \to P(k+1)$, we assume $P(k)$ and try to deduce $P(k+1)$. The statement $P(k)$ says that the sum of the first k positive integers is $\frac{k(k+1)}{2}$. This clearly implies that the sum of the first $k+1$ positive integers is $\frac{k(k+1)}{2} + (k+1)$. Using algebra, we check that this expression coincides with $\frac{(k+1)(k+2)}{2}$ and thereby deduce $P(k+1)$.

A more serious example

We let $P(n)$ be the statement:

$$(x+y)^n = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots y^n,$$

i.e., the binomial theorem for exponent n. The statement $P(1)$ is OK: it says that $(x+y)^1 = x^1 + y^1$. The main work is to show that $P(k)$ implies $P(k+1)$.

We assume $P(k)$ and use this assertion to write

$$(x+y)^{k+1} = (x^k + kx^{k-1}y + \binom{k}{2}x^{k-2}y^2 + \cdots y^k)(x+y).$$

Then one uses the Pascal triangle identity to check that the coefficient of $x^{k+1-i}y^i$ in the product is $\binom{k+1}{i}$, as required.

Some questions

A. How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

B. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people (where the bride and the groom are among these 10 people) if:

1. The bride must be in the picture?
2. both the bride and groom must be in the picture?
3. exactly one of the bride and the groom is in the picture?

C. How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

THIS PAGE INTENTIONALLY LEFT BLANK

For the card problem, consider that the five players have been numbered arbitrarily. There are $\binom{52}{7}$ ways to give the first player her cards, $\binom{45}{7}$ ways to give the second player his cards, and so on. Thus I think that the answer is:

$$\binom{52}{7}\binom{45}{7}\binom{38}{7}\binom{31}{7}\binom{24}{7}.$$

For the wedding problem, it's an extra wrinkle that the people have to be arranged in a row. For each of the sub-problems, if there are N ways to choose the group of 6, there are $6!N$ ways of choosing the group and then ordering the six people in a line.

If the bride must be in the picture, she occupies one place and there are $\binom{9}{5}$ ways to choose the other five people in the group out of the nine remaining people.

If both the bride and groom are in the group, there are (analogously) $\binom{8}{4}$ ways of choosing the other four people.

If either the bride or groom, but not both are in the picture: first decide if it's the bride or groom in the group—there are two ways to do that. Then choose the remaining five people from the eight people who are not from the married couple. Here $N = 2\binom{8}{5}$.

Here's a check: I said that there are $2\binom{8}{5}$ ways to choose the group with the bride or group but not both; also there are $\binom{8}{4}$ ways of choosing the group if both the bride and groom are there. Clearly, there are $\binom{10}{6}$ ways to choose the group if we don't care who exactly is in it and $\binom{8}{6}$ ways to choose the group with neither the bride nor groom. We must therefore have

$$\binom{10}{6} = \binom{8}{6} + \binom{8}{4} + 2\binom{8}{5}.$$

Do we?

$$210 \stackrel{?}{=} 28 + 70 + 2 \cdot 56.$$

Yes, this seems to work.

For the table problem, we can first think about the problem where left–right does matter. We sit down one person—Alice, say—at a seat and then have five choices for Alice's neighbor to the left, four choices for the neighbor to the left of the neighbor, and so on. Thus there are 5! choices for the seating arrangement if we care about handedness. If we don't, we can do a single flip, changing left to right everywhere (and vice versa).

I therefore believe that the answer is $5!/2$.