Methods of Mathematics

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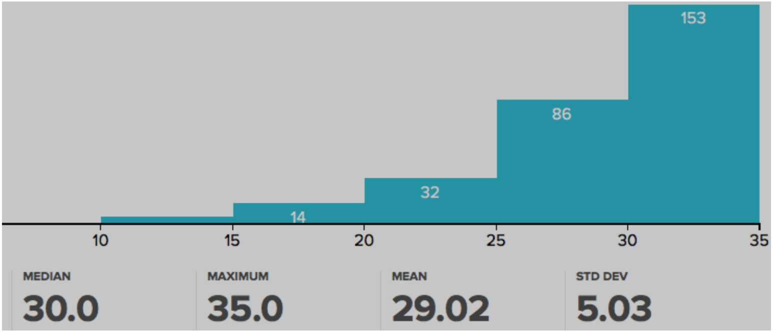
UC Berkeley

Math 10B
February 18, 2016

Midterm 1 scores



These grades are not curved in any way. Do not attempt to match letter grades to the numerical scores.

Ribet Office Hours

Monday 2:10–3:10, 885 Evans

Tuesday 10:30–noon, Student Learning Center

Thursday 10:30–11:30, 885 Evans

The photo shows my office before the renovation this past summer. It was taken in October, 2006.

Indistinguishable boxes

Pop-up lunch tomorrow, Friday, 12:30PM, Faculty Club

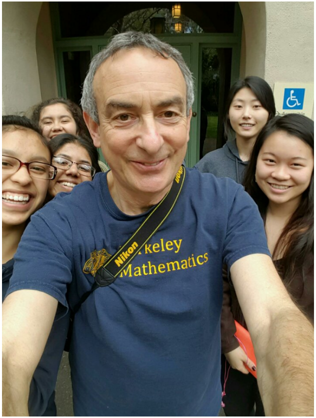
A summary of our course so far.

Upcoming breakfasts

8AM, Monday, February 22 (full)

8AM, Thursday, March 3 (just announced)

Both are at the Faculty Club. Send me email if you'd like to come on March 3.



A Crossroads Lunch

Thursday, February 25 at 12:15PM.

On January 21, I wrote:

> *This lunch is being organized by one of your classmates, who will post on* `piazza` *to try to get a count of how many people want to come.*

Independence

Two events $A, B \subseteq \Omega$ are supposed to be independent if the presence or knowledge of B does not affect the probability of A. A provisional definition would be that A and B are independent if

$$P(A|B) = P(A), \quad \text{i.e., } \frac{P(A \cap B)}{P(B)} = P(A).$$

Multiplying by $P(B)$, we get the official definition that A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

This symmetrical definition holds even if $P(A)$ or $P(B)$ is 0. In that case, both members of the equation are 0 and the two events are independent.

Example (Rosen, and also Example 4.3 of "textbook")

We consider the (eight-element) space of outcomes of three tosses of a fair coin; one such outcome is `HTH`. Determine whether these two events are independent:

> *A*: the outcome is mixed (not `TTT` or `HHH`);
> *B*: there is at most one `T` in the string.

The first event has six of the eight outcomes, so $P(A) = 3/4$. Similarly, there is one outcome with three $\mathtt{T}$'s and three with two $\mathtt{T}$'s; thus B has four elements and $P(B) = 1/2$.

The two are independent if and only if $p(A \cap B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$, i.e., if and only if $A \cap B$ has three elements.

This is true because $A \cap B$ is the set of outcomes with exactly one $\mathtt{T}$, so $A \cap B$ has three elements.

Determine whether these events are independent:

A: Getting all cards of the same suit (generalized flush) in a 5-card hand of poker;

B: Getting a straight (including a straight or royal flush) in a five-card hand of poker.

We learned on February 4 that

$$P(A) = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}.$$

Because a straight can start with any card from ace through 10, there are 10 choices for the five “kinds” in a straight for each the five kinds in the hand, there are four choices of the suit. Hence

$$P(B) = \frac{10 \cdot 4^5}{\binom{52}{5}}.$$

The intersection $A \cap B$ is the set of straight flushes and royal flushes. It has 40 elements: you get to choose the bottom card (10 choices) and the suit (four choices).

Independence would thus mean:

$$40 \cdot \binom{52}{5} \stackrel{?}{=} (10 \cdot 4^5)4 \cdot \binom{13}{5}$$

or

$$\binom{52}{5} \stackrel{?}{=} 4^5 \cdot \binom{13}{5}.$$

According to `Sage`, the left-hand side is 2598960, while the right-hand side is 1317888. The events are not independent.

Sage workshop tonight!!

Sponsored by **MUSA**, the Mathematics Undergraduate Student Association:

February 18, 6–8PM, 1015 Evans
Sage Workshop
featuring Professor William Stein (U. Washington)

Learn to use Sage, a free mathematics software package similar to Mathematica. This workshop is being led by Professor Stein, the inventor of the software. Please bring your laptops.

More indistinguishable boxes

Random variables

A random variable is nothing other than a function on a probability space.

Typically the function is real valued:

$$X : \Omega \to \mathbf{R}.$$

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Here's an interesting and mildly advanced example:

Take a coin that comes up heads with probability p; assume $0 < p < 1$. If $p \neq 1/2$, the coin is biased. We toss the coin repeatedly until it comes up heads and record the sequence of results. For simplicity, assume that it will come up heads—eventually. Then

$$\Omega = \{\, \mathtt{H}, \mathtt{TH}, \mathtt{TTH}, \ldots, \mathtt{T}^{n-1}\mathtt{H}, \ldots \,\}.$$

We have $P(\mathtt{T}^{n-1}\mathtt{H}) = q^{n-1}p$, where $q = 1 - p$ is the probability of getting tails. Sanity check (using geometric series):

$$P(\Omega) = \sum_{n=0}^{\infty} q^{n-1}p = \frac{1}{1-q} \cdot p = 1.$$

A natural random variable is $X(\mathtt{T}^{n-1}\mathtt{H}) := n$; the value of X on a sequence is its length.

For A a subset of $\mathbf{R}$, the condition "$X \in A$" defines an event: the set of all $\omega \in \Omega$ such that $X(\omega) \in A$. We write

$$P(X \in A)$$

for the probability of this event. It's the probability that X is in A.

The values of the random variable X that we just introduced are positive integers. If $A = \{1, 2, 3, 4, 5\}$, then $X \in A$ is the event consisting of `H`, `TH`, `TTH`, `TTTH`, `TTTTH` and has probability

$$p + qp + q^2p + q^3p + q^4p.$$

We can write simply

$$P(X < 6) = p + qp + q^2p + q^3p + q^4p,$$

and we do!

We say that random variables X and Y are independent if

$$P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B)$$

for all subsets A and B of $\mathbf{R}$.