Methods of Mathematics

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UC Berkeley

Math 10B
February 11, 2016

MT #1

The first midterm exam will be next Tuesday, February 16. The exam will be "in class"—in this room.

Every student can bring in one two-sided page of notes (standard-size paper).

The exam "covers" everything through Bayes's Rule: counting of all kinds, probability, conditional probability.

Don't forget to consult the archive of past exams.

Your homework for next week is to work on the problems in the two midterm exams in the archive. Note, however:

- These exams were given on March 4, 2014 and February 26, 2015. These dates are later in the spring semester than February 16. Accordingly, you will see problems that concern concepts that we haven't studied yet (e.g., expected value, random variables. . .).
- One of the exams says "Unless we ask for an actual number, we will accept answers in terms of any combination of [finite] sums, differences, products, quotients, exponents, factorials, $P(n,k)$, $C(n,k)$, $S(n,k)$, and $p_k(n)$." I'm OK with most of this but do *not* want you to leave answers in terms of Stirling numbers or numbers of partitions. When you encounter such numbers, please work out the actual integers that they represent.

Ribet Office Hours

Monday 2:10–3:10, 885 Evans

Tuesday 10:30–noon, Student Learning Center

Thursday 10:30–11:30, 885 Evans

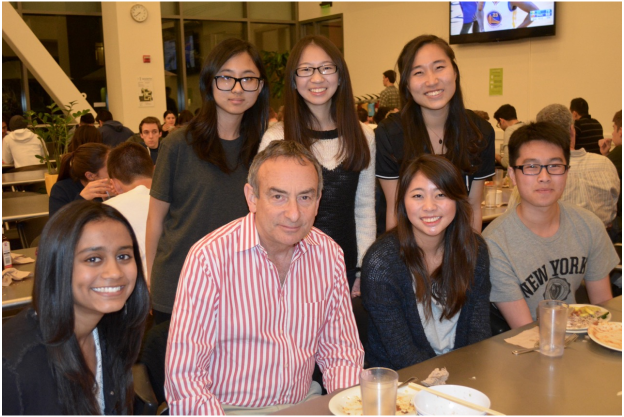
Sorry, there are no office hours on February 15 (UC holiday).

Upcoming breakfasts

8:30AM, Friday, February 12—this one is full

8AM, Monday, February 22

Both are at the Faculty Club. There are still some places available for February 22. Send me email if you'd like to come.



Lunches

There's a pop-up lunch tomorrow at the Faculty Club, 12:30PM. On Fridays, the Club serves a curry, a chowder, fish & chips, and of course its regular salad, sandwich and grill offerings.

Let me know if you'd like to organize a DC or Faculty Club lunch one day in the future. A natural idea is to go to a DC after each SLC office hour.

Stirling was the third son of Archibald Stirling of Garden, Stirling of Keir (Lord Garden, a lord of session). At 18 years of age he went to Balliol College, Oxford, where, chiefly through the influence of the Earl of Mar, he was nominated (1711) one of Bishop Warner's exhibitioners (or Snell exhibitioner) at Balliol. In 1715 he was expelled on account of his correspondence with members of the Keir and Garden families. . . .

Recall that $S(n, k)$ is the number of ways to put n distinguishable people into k indistinguishable offices, subject to the condition that each office gets at least one person.

I feel compelled to explain the recursive formula:

$$S(n+1,k) = kS(n,k) + S(n,k-1),$$

which is on the 95th slide of the “Combinatorics” section of the textbook slides.

Take one of the $n+1$ people; call her Alice. Either Alice gets put in an office with no one else or Alice has officemates. The number $S(n+1k)$ is the sum of two terms, where each term corresponds to one of the two situations.

If Alice gets a private office, there are $k-1$ offices left for the remaining n people. The number of ways of putting the n people into $k-1$ offices (with each office getting at least one person) is $S(n, k-1)$.

If Alice shares an office, put the n other people into the k offices, putting at least one person in each office. There are $S(n, k)$ ways of doing this. Then add Alice to one of the k offices. There are k ways to do this. By the product rule, the number of ways of assigning offices, subject to the restriction that Alice is not alone, is $kS(n, k)$.

Hence $S(n+1, k) = S(n, k-1) + kS(n, k)$, as required.

Example

How many ways can we put 5 distinguishable balls into 3 indistinguishable offices so that each office gets at least one ball? The answer is $S(5,3)$. Now

$$S(5,3) = 3S(4,3) + S(4,2)$$

by the formula. Clearly, $S(4,3) = \binom{4}{2} = 6$ because putting four people into three offices amounts to choosing the two people who share an office. (Every office has to get at least one person, so one office gets two and the other two offices get one.) Also, we saw in class that $S(n,2) = 2^{n-1} - 1$ (with an argument involving Alice again), so $S(4,2) = 7$.

The number $S(4,3)$ is thus $3 \cdot 6 + 7 = 25$. This value is given on page 95 of the combinatorics "textbook" and is also supplied by `Sage` (see `sagemath.org`).

Piazza: “Could you go over the combinatoric proof of #12 and #13 on HW #2?"

#13 is to establish the identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}, \text{ i.e., } \binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}.$$

We imagine choosing, from a class of n students, a committee of k students along with a committee member who will chair the committee. On the left side of the identity, we first choose the committee and then designate one of the k committee members to be chair. On the right side, we first choose the chair by selecting someone from the class. We then fill out the committee by choosing the remaining $k-1$ committee members from the remaining $n-1$ members of the class.

An induction example

Observe that $1 = 1$, $1 + 3 = 4 = 2^2$, $1 + 3 + 5 = 9 = 3^2$,. . . .
This suggests:

For all $n \geq 1$, the sum of the first n odd integers is n^2.

Let $P(n)$ be the statement that the sum of the first n odd integers is n^2. We already have noted that $P(1)$ is true. To prove the statement for all n, we have only to show for all $k \geq 1$:

If $P(k)$ is true, then $P(k + 1)$ is true.

Assume $P(k)$: the sum of the first k odd integers is k^2. The $(k+1)$st odd integer is $2k+1$, so $P(k+1)$ is the statement:

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = (k+1)^2.$$

But

$$1+3+\cdots+(2k-1)+(2k+1) = [1+3+\cdots+(2k-1)]+(2k+1).$$

By $P(k)$, the sum between square brackets is k^2, so the full sum is $k^2 + (2k+1) = (k+1)^2$. In words: the sum of the first $k+1$ odd integers is $(k+1)^2$. This is the statement of $P(k+1)$.

We have seen that $P(k)$ implies $P(k+1)$. By the principle of induction, we conclude that $P(n)$ is true for all n.

Binomial theorem

We want to prove the formula $(x+y)^n = x^n + nx^{n-1}y + \cdots + y^n$ for $n \geq 1$. We can let $P(n)$ be the statement that the formula is true for n and use induction to prove all of the statements.

To begin, we need to check $P(1)$, which is the statement that $(x+y)^1 = x+y$. There's not much to check.

The main task is to prove $P(k) \longrightarrow P(k+1)$. In words, we need to show that if the formula is OK for exponent k then it's OK for exponent $k+1$.

We do this by plugging in (at the right moment) the Pascal triangle formula

$$\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}.$$

The notation gets too hairy for me to do this on slides, though I can try to do it on the crummy white boards.

I'll try to sell the approach by illustrating it on slides for a representative case, say $k = 5$.

So we assume

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and multiply both sides of the equation by $x + y$ to see that

$$(x+y)^6 = (x+y)(x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5).$$

When we expand the product on the right side of the equation, we get 12 terms. Among them are x^6 and y^6, which occur with coefficient 1. The other "monomials" like x^4y^2, occur in two of the 12 terms. For example, x^4y^2 comes about when we multiply x by $10x^3y^2$ and when we multiply y by $5x^4y$. The coefficient of x^4y^2 is then $10 + 5$, i.e., $\binom{5}{2} + \binom{5}{1}$. By the Pascal triangle formula, this sum is $\binom{6}{2}$, which is what it's supposed to be.