Methods of Mathematics

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Math 10B
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We had a breakfast this morning at 8:30AM.

The next breakfast will be April 21 at 8:30AM.

There’s a “pop-up” lunch tomorrow, April 8 at the Faculty Club at 12:30PM. All are welcome.

Actually, I plan to arrive at the Faculty Club soon after noon, so you can come for lunch even if you have a 1PM class.

Next week

I'll be visiting Capitol Hill with other mathematicians on Wednesday (April 13). I'll be leaving campus before noon on Monday and returning (if United Airlines smiles on me) in time for Thursday's class.

- Regular office hours next week are *cancelled*.
- Special office hour in 885 Evans on Friday, April 15: 11AM–12:20PM.
- Pop-up lunch (of course!) at 12:30PM on Friday, April 15.
- Tuesday's class will be taken over by Jason Ferguson—whom many of you know.
- I plan to be back in time for class on Thursday, but Jason will take over again if United misbehaves (a high-probability event).

Linear second-order DEs

Today we will talk about differential equations that look like this:

$$y'' + p(t)y' + q(t)y = r(t).$$

These are second-order linear DEs.

- If the functions $p(t)$ and $q(t)$ are numbers (i.e., constant functions), the equation is said to have *constant coefficients*.
- If $r(t)$ is identically 0, the equation is said to be *homogeneous*.
- If there are no initial conditions, the solution to the DE involves two ("arbitrary") constants. If (for example) $y(0)$ and $y'(0)$ are prescribed, there's a single unique solution.
- These DEs are like totally analogous to the linear recursions that we studied before MT#2.

Following `Dynamics.pdf`, we will limit the discussion to the case of homogeneous equations with constant coefficients. This means that our equation has the form

$$y'' + by' + cy = 0,$$

where b and c are numbers.

To see what's going on, consider the analogous *first order* problem

$$y' + by = 0, \quad \frac{dy}{dt} = -by.$$

The solution is an exponential: $y = Ce^{-bt}$. Sorry if this is mind-bending, but we will consider $-b$ as the root of the characteristic equation

$$\lambda + b = 0$$

gotten by channeling $y' + by$ into a polynomial.

Before moving to the characteristic equation of degree 2, we should focus on a couple of things:

- As I said to students repeatedly when we discussed recursions, it's fruitful to think of the operation

$$y \longmapsto y'' + by' + cy$$

  as a "machine" that takes y as an input and produces $y'' + by' + cy$ as the corresponding output. What makes the situation *linear* is that (1) the sum of two inputs leads to the sum of the corresponding outputs and (2) multiplying the input by a number like 42 multiplies the output by the same number.
- As a result, if y is a solution to $y'' + by' + cy = 0$, then so is $42y$. Also, if y_1 and y_2 are solutions, so is $y_1 + y_2$.

The idea in everything that follows is that we look for two solutions y_1 and y_2 that are not constant multiples of each other. Once we find them, we can assert that

$$C_1 y_1 + C_2 y_2$$

is the *general solution* to the equation. Here C_1 and C_2 are constants; they're like the constants of integration in an integral problem.

Throwback Thursday:

Find a formula for a_n ($n \geq 0$) if

$$a_n = a_{n-1} + 6a_{n-2} \text{ (for } n \geq 2).$$

Today's example:

Solve the DE

$$y'' = y' + 6y.$$

The point is that the DE is a direct analogue of a problem that we've seen.

Our first example

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In the recursion problem, we looked for a solution $a_n = r^n$ and discovered that r needed to satisfy $r^2 = r + 6$. Perhaps pedantically, we said that r needed to be a root of the characteristic polynomial $\lambda^2 - \lambda - 6$.

In today's problem, we look for a solution of the form $y = e^{rt}$; if y is this exponential, then $y' = ry$, $y'' = r^2y$, and thus

$$y'' = y' + 6y \quad \Longleftrightarrow \quad r^2 = r + 6.$$

Now $r^2 - r - 6 = 0$ exactly for $r = -2, 3$. Thus the exponential solutions to $y'' = y' + 6y$ are $y_1 = e^{-2t}$ and $y_2 = e^{3t}$. The general solution to $y'' = y' + 6y$ is $C_1e^{-2t} + C_2e^{3t}$.

Solve the differential equation $y'' = y' + 12y$ subject to the initial conditions $y(0) = 1$, $y'(0) = 18$.

Does the document camera work today?

The answer is perhaps $-2e^{-3t} + 3e^{4t}$.

As Nicholas Kristof has observed:

> *. . . stress among college students is alarmingly high and rising each year with the majority of students feeling consistently anxious, overwhelmed or hopeless. . . .*

That's why we're taking our time with these differential equations examples.

There are two remaining questions that we need to discuss:

1. What happens if the characteristic polynomial has a repeated root?
2. What happens if the characteristic polynomial has two conjugate complex roots?

For the first question, consider this representative example: Solve the differential equation

$$y'' - 2y' + y = 0.$$

The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. There is only one root: 1. The function $y_1 = e^t$ is a solution. How do we get the second fundamental solution?

The answer is that we multiply by t: $y_2 = te^t$ is a second solution.

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To flip the classroom, let's start with the solution and come up with the problem.

The solution is going to be $y = 2e^{-2t} + 3te^{-2t}$.

The characteristic polynomial is then $(\lambda + 2)^2 = \lambda^2 + 4\lambda + 4$, so the differential equation needs to be

$$y'' + 4y' + 4y = 0.$$

We have $y(0) = 2$, so that can be one of our initial conditions. I seem to find

$$y' = -4e^{-2t} + 3(e^{-2t} - 6te^{-2t}) = -e^{-2t} - 6te^{-2t}.$$

If this is right then $y'(0) = -1$; that can be our second initial condition.

The finished problem

Find $y(t)$ if

$$y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

Try this at home and see if you come up with the solution $y = 2e^{-2t} + 3te^{-2t}$.

Complex roots

Here's an example: find the general solution to

$$y'' + y = 0.$$

The characteristic polynomial is $\lambda^2 + 1$, whose roots are $+i, -i$.

What do we do here?

Actually, we know two functions whose second derivatives are their negatives: $\cos t$, $\sin t$.

The general solution in this case is

$$C_1 \cos t + C_2 \sin t.$$

What's going on, secretly or not so secretly, is that the cosine and the sine are expressible in terms of exponentials. Basically by definition, we have

$$e^{it} = \cos t + i \sin t, \quad e^{-it} = \cos t - i \sin t$$

and thus

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}.$$

The fundamental solutions $\cos t$ and $\sin t$ are the real shadows of the complex fundamental solutions e^{it} and e^{-it}.

It probably wouldn't hurt now for me to explain the rule: If the roots of the characteristic polynomial are $\alpha + i\beta$ and $\alpha - i\beta$, then the fundamental solutions to the DE are

$$e^{\alpha t}\cos\beta t, \quad e^{\alpha t}\sin\beta t$$

and the general solution is then

$$y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t.$$

Time to flip the classroom again. Let's imagine that the solution to a problem with initial conditions is going to be

$$y = 2e^t \cos 4t + 3e^t \sin 4t.$$

Then $\alpha = 1$, $\beta = 4$, and the two roots of the characteristic polynomial are $1 \pm 4i$. The characteristic polynomial is

$$\lambda^2 - 2\lambda + 17,$$

so that the DE is $y'' - 2y' + 17y = 0$. We have $y(0) = 2$.

According to Sage, $y' = 14e^t \cos(4t) - 5e^t \sin(4t)$, so that $y'(0) = 14$.

Find $y(t)$ if

$$y'' - 2y' + 17y = 0, \quad y(0) = 2, \quad y'(0) = 14.$$

Wait a day or two and then do this problem at home.