Methods of Mathematics

Kenneth A. Ribet

UC Berkeley

Math 10B
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Breakfasts

The next breakfast will be on Thursday, April 7 at 8:30AM. To sign up, send me email!

There's a "pop-up" lunch on Friday, April 8 (next week) at the Faculty Club at 12:30PM. All are welcome.

Midterm #2

As far as I can tell, 286 students took MT#2. Grades ranged between 5 and 35 (the maximum possible score). The mean was 29.65, and the standard deviation was 5.15.

Here are the 286 grades:

[5, 10, 10.5, 11.5, 12, 16, 16, 16.5, 16.5, 17, 18, 18.5, 19, 19, 19, 19, 19, 19, 20.5, 20.5, 21, 21.5, 21.5, 21.5, 22, 22, 22, 22.5, 23, 23, 23, 23, 23.5, 23.5, 23.5, 24, 24, 24, 24, 24.5, 24.5, 24.5, 24.5, 24.5, 24.5, 24.5, 25.5, 25.5, 25.5, 25.5, 25.5, 25.5, 25.5, 26, 26, 26.5, 26.5, 26.5, 26.5, 27, 27, 27, 27, 27, 27, 27, 27, 27.5, 27.5, 27.5, 27.5, 27.5, 27.5, 28, 28, 28, 28, 28, 28.5, 28.5, 28.5, 28.5, 28.5, 28.5, 28.5, 28.5, 28.5, 28.5, 28.5, 29, 29, 29, 29, 29, 29, 29, 29.5, 29.5, 29.5, 29.5, 29.5, 29.5, 29.5, 29.5, 29.5, 29.5, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30.5, 30.5, 30.5, 30.5, 30.5, 30.5, 30.5, 30.5, 30.5, 30.5, 30.5, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31.5, 31.5, 31.5, 31.5, 31.5, 31.5, 31.5, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32.5, 32.5, 32.5, 32.5, 32.5, 32.5, 32.5, 32.5, 32.5, 32.5, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33.5, 33.5, 33.5, 33.5, 33.5, 33.5, 33.5, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34.5, 34.5, 34.5, 34.5, 34.5, 34.5, 34.5, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35]

There are 291 students for which there are positive scores on at least one midterm. I added everyone's scores for the two MTs, not trying to filter out the people who did not take last Thursday's exam. Nine people got the maximum score of 70 $(= 35 + 35)$. For the sum, the mean was 58.11 and the standard deviation was 9.95.

This class meeting is about *separable* first-order ODEs. The word “separable” means that we can solve the DE by re-writing the equation so that each side of the equation involves only one variable.

Specifically, suppose that the DE can be viewed in the form

$$\frac{dy}{dt} = \frac{f(t)}{g(y)}.$$

Then we write

$$g(y)\,dy = f(t)\,dt, \quad \int g(y)\,dy = \int f(t)\,dt,$$

evaluate the two integrals and walk away with an implicit equation linking y and t.

Here's a quintessential example, where the variables would naturally be called x and y instead of t and y:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

We get

$$\int y\,dy = -\int x\,dx, \quad \ldots \quad x^2 + y^2 = C.$$

We have the equation of a circle (of radius $\sqrt{C}$), and the DE encapsulates the fact that the tangent to the circle at (a, b) (let's say) is perpendicular to the radius of the circle—the segment running from $(0, 0)$ to (a, b). The slope of the radius is b/a, so the slope of the tangent line is $-a/b$.

Basically this entire lecture (course meeting, I prefer to say) is about *examples* of separable equations.

As a first example (beyond the circle), consider

$$\frac{dy}{dt} = \frac{t}{y}.$$

This is like the circle example, except that a sign has changed. (Also, we've gone back to t–y.)

We get *hyperbolas* instead of circles.

Try this example at home.

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For another warmup-type example, here's a separable equation:

$$\frac{dy}{dt} = ky,$$

where k is a constant. This is a linear equation (more precisely, a first-order homogeneous linear ODE with constant coefficients!). We know already that the solution is

$$y = Ce^{kt},$$

where C is a constant.

Again, try this at home: check that you can recover the solution by separating and integrating:

And another home exercise: look through the slides of the class meetings in March to see if you can spot other DEs that can be solved by separating variables. How many of those were solved in class by this method?

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"Solve the IVP

$$y' = \frac{2ty^2}{1+t^2}, \quad y(0) = y_0.$$

Discuss what happens when y_0 is 0, when it's positive, when it's negative."

This example appears as Exericse 12 (p. 27) of David J. Logan's book A First Course in Differential Equations. If you can snag a copy, try doing the other exericses in this section (which is on separable equations).

The formal solution seems to be

$$-\frac{1}{y} = \ln(1+t^2) + C.$$

If $y_0 \neq 0$, we can plug in $t = 0$ and get that $C = -\frac{1}{y_0}$. If $y_0 = 0$, there is no C that makes the equation work; on the other hand $y = 0$ is clearly a solution.

With the assumption $y_0 \neq 0$, we get

$$y = \frac{1}{\frac{1}{y_0} - \ln(1 + t^2)}.$$

The denominator is 0 exactly when $1 + t^2 = e^{1/y_0}$. The left-hand side of this equation is always ≥ 1. If y_0 is negative, the right-hand side e^{1/y_0} is less than 1; hence the denominator is non-zero for all t. If y_0 is positive, the denominator is non-zero for $|t| < \sqrt{e^{1/y_0} - 1}$ but vanishes for $t = \pm\sqrt{e^{1/y_0} - 1}$. In this case, there is a solution $y(t)$ for t inside the interval $(-\sqrt{e^{1/y_0} - 1}, +\sqrt{e^{1/y_0} - 1})$; this solution blows up at the endpoints of the interval.

Logistic equation

The online "textbook" has a paragraph or two motivating the DE

$$p' = rp(1 - \frac{p}{k}),$$

where r and k are positive constants and p is a function of t (time). The symbol "p" refers to population; k is the maximum carrying capacity of the environment. Also, r is a "growth rate." The idea is that p cannot ever get bigger than k. In fact, the solution to the DE shows that $p \to k$ as $t \to \infty$ and that p is always smaller than k.

Our aim will be to solve this equation, which means to write down the formula expressing p in terms of t.

Actually, the "textbook" has a good explanation of the derivation of the solution, so I'll just summarize the result on these slides and then do the derivation on paper (document camera).

The first step is to note that there are constant solutions $p = 0$ and $p = k$ where both sides of the DE are identically 0.

If we are not dealing with one of these two, then we separate and integrate, which leads to a natural log. (The integration requires a simple partial fraction decomposition.) Exponentiating leads to

$$\frac{p}{k - p} = Ke^{rt},$$

where K is a non-zero constant. If we relax the requirement that K be non-zero, then we also pick up the solution $p = 0$ by this formula. You might possibly want the solution $p = k$ to correspond to $K = \infty$. I'm not sure that I encourage this.

A little algebra gives the formula

$$p = k\,\frac{Ke^{rt}}{1 + Ke^{rt}}.$$

Unless I'm missing something, we can check that $p \to k$ as $t \to \infty$ if K is non-zero and r is positive.

In the online file `Dynamics.pdf`, there's an obvious typo in a lot of equations on page 78: the constant k has disappeared, meaning that it's implicitly being set to 1. The normalization $k = 1$ is pretty standard in the subject but was not adopted by the "textbook." See Wikipedia, where there's talk of the "standard logistic function."

For comparison with Wikipedia, it's maybe worth noting that

$$\frac{Ke^{rt}}{1+Ke^{rt}} = \frac{1}{1+1/(Ke^{rt})}$$

and that we can write $1/(Ke^{rt}) = e^{-r(t-t_0)}$ for a suitable choice of t_0. If $k = 1$, the function $p(t)$ is then $\dfrac{1}{1+e^{-r(t-t_0)}}$.

Note that this ('sigmoid') function $\to 0$ when $t \to -\infty$ and $\to 1$ when $t \to \infty$. There's a nice graph on the aforementioned Wikipedia page.

I'll close by quoting a DE exercise from the calculus book that I used in high school. "Calculus and Analytic Geometry," 3rd ed., by George B. Thomas:

$$\ln x \frac{dx}{dy} = \frac{x}{y}.$$

In this problem, x and y are understood to be functions of each other. Odds are that we will emerge with an equation that links them together, rather than a formula of the form $y = f(x)$ or $x = g(y)$.

We write

$$\frac{1}{x} \ln x \, dx = \frac{1}{y} \, dy$$

and then integrate both sides:

$$\frac{1}{x}(\ln x)^2 = \ln |y| + C.$$

The answer in the back of the book is

$$\frac{1}{2}(\ln|x|)^2 = \ln|y| + C.$$

It's not clear to me where the "$|x|$" comes from: the problem as stated has $\ln x$, which is defined only for positive x. Thus the values of x are required implicitly to be positive.

I don't see why we don't exponentiate both sides, getting

$$y = Ke^{(\ln x)^2/x},$$

where K is a non-zero constant. Here we see at least that y is written as a function of x.