Methods of Mathematics

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Math 10B
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There is a new version of the online “textbook” file `Matrix_Algebra.pdf`.

The next breakfast will be two days from today, April 21 at 8:30AM. There is still *lots* of room. Send me email to sign up—and tell your friends to do the same.

There’s a “pop-up” lunch on Friday, April 22 at the Faculty Club at 12:30PM. All are welcome.

Regular office hours this week!

Dear Kenneth A. RIBET,

Evaluations for your course(s) have opened to students.

Although students received an invitation email and reminders along the way, previous research demonstrates that a personal reminder from the instructor and an explanation of how evaluations are used to inform your teaching can make a positive impact on response rate and quality.

Course Evaluations
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Four remaining lectures:

- Today: Gaussian elimination
- This Thursday: Eigenvalues and eigenvectors
- Next Tuesday: Least squares, linear regression
- Next Thursday: Dynamic programming

RRR week and exam week follow.

Our exam: Monday, May 9, 11:30AM–2:30PM.

Gaussian elimination is actually the technique that most of you already know for solving systems of linear equations. The best way to discuss the technique is to start with a 3×3 example:

$$\begin{aligned} 3x - y + 2z &= 7 \\ x + 2y - 5z &= -10 \\ -x - y + 3z &= 6. \end{aligned}$$

Of course, one can proceed in lots of ways, but I propose to add the third equation to the second, to add 3 times the third equation to the first, to multiply the bottom equation by -1 and to move the bottom equation to the top:

$$\begin{aligned} x + y - 3z &= -6 \\ -4y + 11z &= 25 \\ y - 2z &= -4. \end{aligned}$$

Now I'll add 4 times the bottom equation to the middle equation and then swap the two bottom equations:

$$\begin{aligned} x + y - 3z &= -6 \\ y - 2z &= -4 \\ 3z &= 9. \end{aligned}$$

I can't resist dividing the new bottom equation by 3:

$$\begin{aligned} x + y - 3z &= -6 \\ y - 2z &= -4 \\ z &= 3. \end{aligned}$$

The last equation now tells us that $z = 3$; then the middle equation yields that $y - 6 = -4$, so $y = 2$. The first equation then reads $x + 2 - 9 = -6$, so $x = 1$. Thus $(x, y, z) = (1, 2, 3)$.

To get from the first set of equations to the last set of equations, we performed operations of this kind:

- Add a non-zero multiple of one equation to another equation;
- Multiply one of the equations by a non-zero constant (e.g., $1/3$);
- Re-order the equations (which we can do by a series of exchanges if we want).

The *elementary row operations* are the additions and multiplications in the first two items and the exchanges in the third item. We can effect any desired re-ordering through a sequence of exchanges, as I said.

Optimally, the last equation then tells us the value of the last variable; then the next-to-last equation tells us the value of the next-to-last variable, etc., etc. Finding the actual values is called "back substitution."

If you're a professional (or a computer), you don't really need all the symbols that are present in the initial system

$$
\begin{aligned}
3x - y + 2z &= 7 \\
x + 2y - 5z &= -10 \\
-x - y + 3z &= 6.
\end{aligned}
$$

You can just write down the *augmented matrix*

$$\left(\begin{array}{rrr|r} 3 & -1 & 2 & 7 \\ 1 & 2 & -5 & -10 \\ -1 & -1 & 3 & 6 \end{array}\right),$$

manipulating it by applying elementary row operations you find the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 1 & -3 & -6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right),$$

which makes it apparent how to solve the original system.

If $A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & -5 \\ -1 & -1 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ -10 \\ 6 \end{pmatrix}$, then $A^{-1}b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. In other words, by doing Gaussian elimination, we've computed A^{-1} times b. However, b is a completely arbitrary set of numbers; the only virtue in b is that I chose it so that the components of $A^{-1}b$ didn't involve fractions.

Now here's the thing: instead of taking $b = \begin{pmatrix} 7 \\ -10 \\ 6 \end{pmatrix}$, we could take $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then we'd compute $A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which is actually the first column of A^{-1}. Similarly, using $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we could get the other two columns of A^{-1} and thereby compute the inverse of A.

The professional method is to do Gaussian elimination on the augmented matrix

$$\left(\begin{array}{rrr|rrr} 3 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & -5 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right),$$

using the same operations as when we solved the system of equations

$$\begin{aligned} 3x - y + 2z &= 7 \\ x + 2y - 5z &= -10 \\ -x - y + 3z &= 6. \end{aligned}$$

I ended up with

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 & 4 & 7 \end{array}\right)$$

and computed the three columns of A^{-1} by using back substitution and the three columns of the right-hand side of this new matrix. For the record,

$$A^{-1} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 11/3 & 17/3 \\ 1/3 & 4/3 & 7/3 \end{pmatrix}.$$

It's easy to guess that $\det A = 3$; this is correct.

The good way to proceed is to continue operating on

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 & 4 & 7 \end{array}\right)$$

until the left-hand side becomes the identity 3×3 matrix. The right-hand side then displays A^{-1}. After bumbling around at home, I succeeded in doing this and hope to duplicate my work on the document camera.

As a first step, we divide the bottom row by 3, getting as the new bottom row

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1/3 & 4/3 & 7/3 \end{array}\right),$$

whose right-hand side is the bottom row of A^{-1}.


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To conclude the story, I should explain what happens when there are infinitely many solutions to a system, and when there are no solutions to the system. Consider

$$\begin{aligned} 3x - y + 2z &= 7 \\ x + 2y - 5z &= -10 \\ 4x + y - 3z &= -3. \end{aligned}$$

The third equation is the sum of the first two—it just comes along for the ride. There are really two equations in three unknowns, so there's every chance of having an infinite number of solutions. We have to imagine that someone hands us this system and that we try to solve it by Gaussian elimination.

I did this at home and got into a situation where one of the three equations was literally repeated. Then I was left with

$$\begin{aligned} -7y + 17z &= 37 \\ 3x - y + 2z &= 7. \end{aligned}$$

The interpretation is that z can be *anything* and that y can be computed from z via the first equation; then the value of x is found from the second equation. If $z = t$ (a "parameter"), then

$$y = \frac{-37 + 17t}{7}, \quad x = \frac{4 + t}{7}.$$

If, for example, $t = 3$, then $x = 1$ and $y = 2$. If $t = -4$, $x = 0$ and $y = -15$.

Now consider

$$\begin{aligned} 3x - y + 2z &= 7 \\ x + 2y - 5z &= -10 \\ 4x + y - 3z &= +3. \end{aligned}$$

I've changed the sign on the bottom equation. When we start Gaussian elimination, we would naturally subtract 3 times the second equation from the first equation and 4 times the second equation from the third equation.

Doing this led me to

$$\begin{aligned} x + 2y - 5z &= -10 \\ -7y + 17z &= 37 \\ -7y + 17z &= 43. \end{aligned}$$

Subtracting the second equation from the third, we get $0 = 6$, which is trouble.