Derivatives: definition and first properties

Math 10A

September 7, 2017

Ribet office hours

- Mondays, 1:45–3PM, 885 Evans
- Wednesdays, 10:30–11:45AM, SLC

Yesterday at the SLC

Lunch

The next pop-in lunch at the Faculty Club will be tomorrow, September 8 at high noon.

Breakfasts

The next breakfasts are as follows:

- September 11 (full)
- September 15 (some spaces available)
- September 18 (full)
- September 20 (full)

All breakfasts this semester are at 8AM. They are held in the Kerr Dining Room of the Faculty Club.

Alternative Facts, Revisited

Schreiber defines

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

The next slide shows that

$$\exp'(0) = 1, \text{ where } \exp(x) = e^x.$$

Let $t = \exp'(0)$, so that $\frac{1}{t} = \ln'(1)$ (as explained on Tuesday).

Since $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$,

$$\begin{aligned} 1 &= \ln\left(\lim_{n\to\infty}(1+\frac{1}{n})^n\right) = \lim_{n\to\infty}\ln\left((1+\frac{1}{n})^n\right) \\ &= \lim_{n\to\infty} n\cdot\ln(1+\frac{1}{n}) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} \\ &= \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \ln'(1) = 1/t. \end{aligned}$$

Hence $t = 1$.

Derivative of exp

If $f(x) = e^x$, we learned on Tuesday that

$$f'(a) = e^a f'(0) \text{ for all real numbers } a.$$

We have just found out that $f'(0) = 1$, so

$$f'(a) = e^a = f(a).$$

In other words, the exponential function is *its own derivative*!

Derivative of ln

The slope of the line tangent to the graph of $y = e^x$ at (a, e^a) is e^a. That's another way of saying that e^x is its own derivative.

Now take the graph of $y = \ln x$. At the point (e^a, a) on this graph, the slope is $1/e^a$. (Reversing the roles of x and y leads to reciprocals.)

Let $b = e^a$, $a = \ln b$. At the point $(b, \ln b)$ on the graph, the slope is $1/b$. Thus $\ln'(b) = 1/b$ for all b.

The derivative of $\ln x$ is $\dfrac{1}{x}$.

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$$\text{The derivative of } \ln x \text{ is } \frac{1}{x}.$$

Next steps

We can differentiate polynomials like $x^{17} - 10x^2 + 37x - 3$, the exponential function, the natural log function. We're doing well. Suppose we want to increase our toolkit so that we can differentiate things like

$$\frac{x^3 \sin(1/x)}{\tan(\ln x) + \sqrt{x^3 + 1}};$$

how do we proceed?

We need to do two things:

1. Make sure that we can calculate the derivatives of trigonometric functions (sin, cos, tan,...).
2. Figure out how to differentiate complicated expressions that are made up of standard "building blocks" that we already understand.

In particular, we need to be able to differentiate the product of two functions whose derivatives we know. We need to differentiate quotients (same sort of problem). We need to differentiate *composite* functions—functions made by chaining known functions together (example: $\sin(1/x)$).

There are a whole bunch of formulas to understand and to learn.

We'll take them one at a time.

But first a common notation: if $y = f(x)$, the derivative of y with respect to x is often denoted $\frac{dy}{dx}$. We think of $\frac{d}{dx}$ as an "operator" that we apply to a function to get its derivative.

For example, people write

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

or

$$\frac{d(\ln x)}{dx} = \frac{1}{x}.$$

Derivative of a product

If f and g are two functions with derivatives f' and g', then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x).$$

This product rule of the seventeenth century is often called "Leibniz's rule."

It's much better to understand why this is true than to think of it as a formula to be memorized. The derivative of $f(x)g(x)$ at $x = a$ is the limit

$$\lim_{b \to a} \frac{f(b)g(b) - f(a)g(a)}{b - a}.$$

Rewrite the fraction as the sum

$$\frac{f(b)g(b) - f(a)g(b)}{b - a} + \frac{f(a)g(b) - f(a)g(a)}{b - a}$$

by subtracting and then adding the same quantity $f(a)g(b)$. In the first summand, $g(b)$ is a common factor; in the second, $f(a)$ is a common factor.

Thus the derivative is

$$\lim_{b \to a} g(b) \frac{f(b) - f(a)}{b - a} + f(a) \lim_{b \to a} \frac{g(b) - g(a)}{b - a}.$$

The first term is $g(a)f'(a)$; the second is $f(a)g'(a)$.

Here we've made silent use of the fact that $\lim_{b \to a} g(b) = g(a)$, i.e., that g is *continuous* at a. This is OK because differentiable functions are continuous. I'll even explain why if you like. . . .