Derivatives: definition and first properties

Math 10A

September 5, 2017

Extra Office Hour

Today, 11AM–noon, 885 Evans.

Lunch to follow if you want.

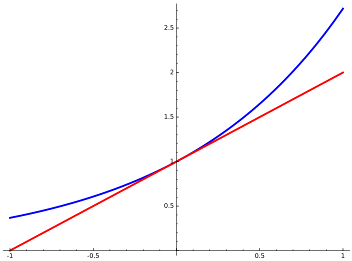
DSP

If you're a DSP student and need exam accommodations, please contact me if you haven't done that already.

What is a derivative?

If a function f is defined on some interval and a is inside the interval, we define the *derivative of* $f(x)$ *at* $x = a$.

This number, denoted $f'(a)$, is the slope of the line that kisses the curve $y = f(x)$ at the point $(a, f(a))$.



The derivative of e^x at $x = 0$ is 1.

Experts say that the line is *tangent* to the curve and don't say much about kissing.

The tangent line is determined by its slope (and the fact that it passes through $(a, f(a))$. The slope is a *limit* of the slopes of *secant lines* connecting $(a, f(a))$ to $(b, f(b))$, where b is near a and approaches a.

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

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A first example, involving sin

The derivative of $\sin x$ at $x = 0$:

$$\lim_{h \to 0} \frac{\sin h}{h}.$$

This limit is 1, as we saw last week; we can write $\sin' 0 = 1$.

We'll look at more examples to get a feel for what happens.

Take $f(x) = x^2$ and $a = 3$:

$$f'(3) = \lim_{b \to 3} \frac{b^2 - 3^2}{b - 3}$$
$$= \lim_{b \to 3} (b + 3) = 3 + 3 = 6.$$

If a were 17 instead of 3, we'd end up with

$$f'(17) = 17 + 17 = 34.$$

In general, $f'(a) = 2a$.

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If a were 17 instead of 3, we'd end up with

$$f'(17) = 17 + 17 = 34.$$

In general, $f'(a) = 2a$.

This example is fairly typical: to compute the limit of a fraction, you can often simplify the fraction and then figure out the limit.

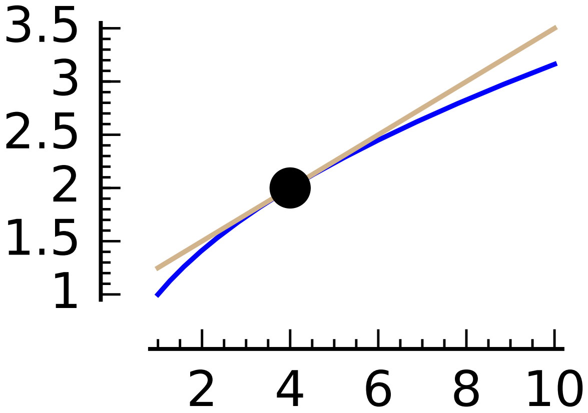
We'll do a few more examples in this direction and then empower you to use the general rule for power functions—functions of the form $y = x^n$, where the exponent n is pretty much any real number.

For example, $y = \sqrt{x}$. If $f(x) = \sqrt{x}$, then:

$$
\begin{aligned}
f'(a) &= \lim_{b \to a} \frac{\sqrt{b} - \sqrt{a}}{b - a} \\
&= \lim_{b \to a} \frac{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})}{(b - a)(\sqrt{b} + \sqrt{a})} \\
&= \lim_{b \to a} \frac{1}{\sqrt{b} + \sqrt{a}} = \frac{1}{2\sqrt{a}} \\
&= \frac{1}{2} a^{-1/2}.
\end{aligned}
$$

"The derivative of $x^{1/2}$ is $\frac{1}{2}x^{-1/2}$."

What is the derivative of $y = \sqrt{x}$ at $x = 4$?



The derivative is $\frac{1}{2} \cdot \frac{1}{\sqrt{4}} = 1/4$, although the aspect ratio makes the derivate look like 1 in the picture. #fakenews

The derivative of $y = x^m$ at $x = a$ is ______?

Try $m = -1/2$, $y = \dfrac{1}{\sqrt{x}}$:

$$\begin{aligned} f'(a) &= \lim_{b \to a} \frac{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}{b - a} \\ &= \lim_{b \to a} \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a}\sqrt{b}(b - a)} \\ &= -\lim_{b \to a} \frac{1}{\sqrt{a}\sqrt{b}} \cdot \frac{\sqrt{b} - \sqrt{a}}{b - a} \\ &= \frac{-1}{a} \cdot \frac{1}{2} a^{-1/2} = -\frac{1}{2} a^{-3/2}. \end{aligned}$$

We're assuming, of course, that a and b are positive in this discussion.

The rule

If $f(x) = x^m$, then

$$f'(a) = ma^{m-1}.$$

"You multiply by the exponent and decrease the exponent by 1."

If $m < 1$, you need to assume that a is non-zero.

If $m = 1$, the derivative is 1 for all a, so the symbolic expression a^0 should be interpreted as 1.

Sums and multiples

- The derivative of a sum of two functions is the sum of the derivatives of the two functions.
- The derivative of 37 times a function is 37 times the derivative of the function. (The number 37 was chosen just for illustration.)

The derivative of $x^3 - 3x^2 + 10x + 1$ at $x = a$ is

$$3a^2 - 6a + 10 + 0.$$

"The derivative of $x^3 - 3x^2 + 10x + 1$ is $3x^2 - 6x + 10$."

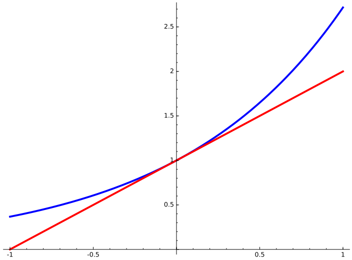
The derivative of $f(x) = e^x$ at $x = a$ is

$$\lim_{h\to 0} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h\to 0} \frac{e^h - 1}{h} = e^a f'(0).$$

The number $f'(0)$ in this case is equal to 1.

[We used the formula $e^{a+h} = e^a e^h$ in the displayed formula above.]

Why is it true that the number $f'(0)$ in this case is equal to 1?



This beautiful picture was on the first slide after the title page!

Schreiber, page 204: “Although it is beyond the scope of this book, it turns out...." Sheesh!

Alternative Facts

It's a much better idea to define e so that e^x has derivative 1 at $x = 0$. Let's pretend that Schreiber did that.

Then the natural log function ln would have derivative 1 at $x = 1$. (I'll explain this on the board.)

Can we now compute $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and see that it's e?

Take ln of the limit, which is

$$\lim_{n\to\infty} n \cdot \ln\left(1 + \frac{1}{n}\right)$$

and hope that we get 1 (the natural log of e). This limit is

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{1/n} = \lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h}.$$

This last limit is *by definition* the derivative of ln at 1, which we'd know to be 1.

Booksellers

Josh Adams, who wrote to me on August 29
Avani Kelekar who wrote on September 2