Yet Another Derivatives Discussion

Math 10A

September 12, 2017

Two breakfasts and a lunch

The fourth Math 10A breakfast was held yesterday at 8AM:

The fifth breakfast will be held on Friday at the same time.

In addition, there'll be a pop-in lunch (featuring chicken vindaloo) on Friday at noon.

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Announcing a new breakfast on Monday, September 25 at 8AM. Send email if you'd like to come.

My next office hour will be at 10:30AM on Wednesday in the SLC.

Derivative of sin

To calculate the derivative of sin at $x = a$, we can think of the secant line connecting $(a - h, \sin(a - h))$ and $(a + h, \sin(a + h))$ for small $h \to 0$. The change in x from $a - h$ to $a + h$ is $2h$, so we have to calculate

$$\lim_{h \to 0} \frac{\sin(a+h) - \sin(a-h)}{2h}.$$

It helps to remember these high school formulas:

$$\sin(a+h) = \sin(a)\cos(h) + \cos(a)\sin(h),$$
$$\sin(a-h) = \sin(a)\cos(h) - \cos(a)\sin(h).$$

The second follows from the first because $\cos(-h) = \cos(h)$, $\sin(-h) = -\sin(h)$.

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Subtraction gives

$$\sin(a+h) - \sin(a-h) = 2\cos(a)\sin(h),$$

so that

$$\begin{aligned}
\sin'(a) &= \lim_{h\to 0} \frac{2\cos(a)\sin(h)}{2h} \\
&= \cos(a) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \cos(a)\cdot 1.
\end{aligned}$$

We have now seen that the derivative of sin is cos.

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Other trigonometric functions

As explained last Thursday,

$$\frac{d}{dx}\cos(x) = -\sin\,.$$

We can see that in the same way as in calculating the derivative of sin. Alternatively, we can use the Chain Rule (soon to be explained) and the formula $\cos(x) = \sin(\frac{\pi}{2} - x)$ to express cosine in terms of sine.

Then we get the derivatives of tan, sec, the cotangent function, etc.

You'll probably want to familiarize yourself with the formulas for the derivatives.

Everyone gets a sheet of notes

You don't need to *memorize* the formulas for the derivatives.

For the exams (two midterms, one final), you can bring in one two-sided sheet of notes. On the sheet, you can write down all sorts of formulas. Thus there's no need to memorize things that don't want to stick in your brain.

Chain rule

We're close to the end learning how to take derivatives, but we need one more formula and one simple technique.

The formula is called the *chain rule*. It tells you how to differentiate a "function of a function," namely something like $f(g(x))$:

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x))g'(x).$$

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Two examples

To find $\frac{d}{dx}(sin(x^3))$, we apply the formula and get

$$\frac{d}{dx}(sin(x^3)) = cos(x^3)3x^2.$$

Similarly,

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2}2x.$$

Note that e^{x^2} means "e raised to the power x^2."

How to remember the formula without pain

To find $\dfrac{d}{dx}(f(g(x)))$, let $y = f(g(x))$ and let $u = g(x)$. Then $y = f(u)$ and $f'(u) = f'(g(x))$ can be thought of as $\dfrac{dy}{du}$. Also, $g'(x) = \dfrac{du}{dx}$. Hence the chain rule reads

$$\frac{d}{dx}(f(g(x)) = f'(g(x))\, g'(x)$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The chain rule gives you the right to perform a symbolic cancellation of *du*.

Find the derivative of $\sqrt{1+x^2}$ (with respect to x).

Let $y = \sqrt{1+x^2}$, $u = 1 + x^2$. Then $y = \sqrt{u}$, so that

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{1+x^2}}.$$

Also $\frac{du}{dx} = 2x$. Thus

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}.$$

That was a typical kind of problem.

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Implicit differentiation

Find the equation of the line tangent to the curve $2x^2 + 3y^2 = 66$ at the point $(-3, -4)$.

We could solve for y and differentiate directly, but the computation is a lot easier if we differentiate *implicitly*. Because y is some function of x near $(-3, -4)$ we can write $\frac{dy}{dx}$ for the derivative of this function and see what we get.

In other words, we blithely differentiate both sides of the equation $2x^2 + 3y^2 = 66$ and just keep track of what happens.

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Start with

$$2x^2 + 3y^2 = 66$$

and differentiate both sides with respect to x:

$$4x + 6y\frac{dy}{dx} = 0.$$

Note that $\frac{d}{dx}(3y^2) = 6y\frac{dy}{dx}$ by the chain rule!

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-4x}{6y} = -\frac{2x}{3y}.$$

At $(-3, -4)$, $-\frac{2x}{3y} = -6/12 = -1/2$. Hence the tangent line to the curve at $(-3, -4)$ has slope $-1/2$. Its equation is then

$$\frac{y+4}{x+3} = -1/2, \quad y = -4 - (x+3)/2.$$

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