Integrals, areas, Riemann sums

Math 10A

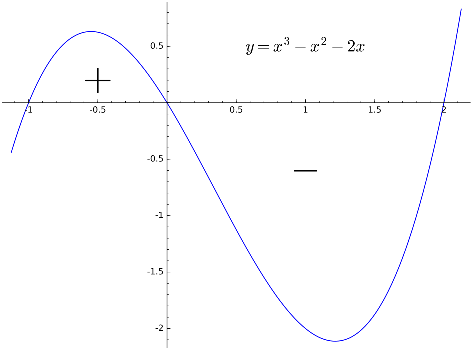
October 5, 2017

We had the ninth Math 10A breakfast yesterday morning:

There are still lots of slots available Breakfast #10, next Monday (October 9) at 9AM.

Signed area

Find the area enclosed between the cubic $y = x^3 - x^2 - 2x$ and the x-axis from $x = -1$ to $x = 2$.



Pro-tip: the answer is *not* $\int_{-1}^{2} (x^3 - x^2 - 2x)\, dx$!

The integral $\int_{-1}^{2} (x^3 - x^2 - 2x)\,dx$ adds the "positive" area between -1 and 0 to the "negative" area between 0 and 2, thereby getting the incorrect answer $-9/4$.

The correct answer may be written as $\int_{-1}^{2} |x^3 - x^2 - 2x|\,dx$, but that's not especially helpful because we can't integrate absolute values very well.

The best move is to divide the region of integration into the two segments $[-1, 0]$ and $[0, 2]$.

The positive area then becomes

$$\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx - \int_{0}^{2}(x^3 - x^2 - 2x)\,dx$$
$$= 2F(0) - F(-1) - F(2),$$

where F is an antiderivative of $x^3 - x^2 - 2x$, say $F(x) = x^4/4 - x^3/3 - x^2$. With this choice,

$$F(0) = 0, \quad F(-1) = -5/12, \quad F(2) = -8/3.$$

The total area is

$$2F(0) - F(-1) - F(2) = 5/12 + 8/3 = 37/12.$$

How is area actually defined?

Area is a limit of Riemann sums.

To define $\int_a^b f(x)\,dx$: Choose an integer $n \geq 1$ and divide up $[a, b]$ into n equal pieces

$$[a, a+\frac{b-a}{n}], [a+\frac{b-a}{n}, a+2\cdot\frac{b-a}{n}], \ldots [a+(n-1)\cdot\frac{b-a}{n}, b].$$

Each interval has length $\Delta x = (b-a)/n$. The last endpoint b is $a + n \cdot \dfrac{b-a}{n}$.

There are n intervals. Choose x_1 in the first interval, x_2 in the second interval, etc. The Riemann sum attached to these choices is

$$\frac{b-a}{n}\left(f(x_1)+f(x_2)+\cdots+f(x_n)\right).$$

It's the sum of the areas of n rectangles, each having base $\frac{b-a}{n}$. The heights of the rectangles are $f(x_1)$, $f(x_2)$,..., $f(x_n)$.

The Riemann sum is an approximation to the true area. As $n \to \infty$ and the rectangles get thinner, the approximation gets better and better.

The integral $\int_a^b f(x)\,dx$ is the *limit* of the Riemann sums as $n \to \infty$.

The choices of the points x_i in the intervals is irrelevant. It is most common to take the x_i to be the left- or the right-endpoints of the intervals. One could take them to be in the middle of the intervals.

A simple example

To see $\int_0^1 x\,dx$ as a limit of Riemann sums, divide the interval $[0,1]$ into n equal pieces and let the x_i be the right endpoints of the resulting small intervals:

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n}.$$

The Riemann sum is

$$\begin{aligned}\frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n}\right) &= \frac{1}{n^2}(1 + 2 + 3 + \cdots + n)\\ &= \frac{1}{n^2}\frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.\end{aligned}$$

We used that the arithmetic progression $1 + 2 + \cdots + n$ has sum $\frac{n(n+1)}{2}$, a fact that can be explained easily on the document camera.

This is a completely silly way to find the area of a right triangle with base and height both equal to 1.

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To find $\int_0^1 x^2\,dx$, we'd need to know the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

There are similar formulas for the sum of the kth powers of the first n integers, though knowing the full formulas is not necessary for computing the limits of the Riemann sums.

The Fundamental Theorem of Calculus just tells us that $\int_0^1 x^k\,dx = \frac{1}{k+1}$ for $k \geq 1$, so we don't need explicit formulas to compute integrals.

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A 2016 midterm problem

Express $\int_{-1}^{1} \cos x \, dx$ *as a limit of Riemann sums.*

This problem came from the textbook: it's #12 of §5.3 with the absolute value signs removed (to make the problem easier).

There is no single correct answer because the user (you) gets to choose the points x_i.

Divide the interval $[-1, 1]$ into n equal segments and use left endpoints for the x_i. The intervals have length $\frac{2}{n}$, so the Riemann sum with n pieces is

$$\frac{2}{n} \sum_{i=0}^{n-1} \cos\left(-1 + \frac{2i}{n}\right).$$

The integral is the limit of this sum as n approaches ∞.

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Express $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{2i}{n}\right)\left(\frac{2}{n}\right)$ *in the form* $\int_0^1 f(x)\,dx$.

This is problem 4 of §5.3 of the textbook.

We can write the expression before taking the limit as

$$\frac{2}{n}\sum_{i=1}^{n}\left(1 - \frac{2i}{n}\right).$$

This looks like a Riemann sum for an interval of integration of length 2. Because you're asked to shoehorn the problem into an integral $\int_0^1 \cdots\, dx$, the problem is challenging.

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It helps to write the sum as

$$\frac{1}{n}\sum_{i=1}^{n} 2\left(1-\frac{2i}{n}\right),$$

to make the Δx term into the expected $\frac{1}{n}$.

To have

$$\frac{1}{n}\sum_{i=1}^{n} 2\left(1-\frac{2i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n}),$$

we take $f(x) = 2(1-2x) = 2-4x$.

Yet more challenging

Evaluate the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{2i}{n}\right)\left(\frac{2}{n}\right)$.

Once we write the limit as $\int_0^1 (2 - 4x)\, dx$, we can evaluate the integral and be finished. The integral is

$$(2x - 2x^2)\Big]_0^1 = 0.$$

This is plausible because the terms $\left(1 - \frac{2i}{n}\right)$ are positive for i small and negative for i near n. The first term in the parentheses is $1 - \frac{2}{n} > 0$ (for $n > 2$) and the last term is -1. Apparently there's cancellation!

Substitution

The chain rule states:

$$\frac{d}{dx}(F(u)) = F'(u)\frac{du}{dx}.$$

Thus, in the world of antiderivatives:

$$\int F'(u)\frac{du}{dx}\,dx = F(u) + C.$$

It is natural to cancel the two factors *dx* and write this as

$$\int F'(u)\,du = F(u) + C.$$

Further, if F' is given as a function f and F is introduced as an antiderivative of f, then we have the formula

$$\int f(u)\,du = F(u) + C,$$

where F is an antiderivative of f.

This makes sense after we do examples: Evaluate

$$\int \cos(x^2) 2x \, dx.$$

It's up to us to introduce u, so we set

$$u = x^2, \quad \frac{du}{dx} = 2x, \quad du = 2x \, dx.$$

In terms of u, the integral to be evaluated is

$$\int \cos u \, du = \sin(u) + C = \sin(x^2) + C.$$

In other words, we computed $\sin(x^2)$ as an antiderivative of $2x \cos(x^2)$.

Conclusion: if you need to evaluate an indefinite integral and can't see the antiderivative immediately, try to make the integrate simpler by a judicious substituion $u = \cdots$ (some function of x).

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It would be more common to encounter the indefinite integral

$$\int \cos(x^2)x\,dx;$$

the factor “2” has disappeared. Again, we set $u = x^2$ and write $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$. In terms of u, the integral becomes

$$\int \cos u\,\frac{1}{2}du = \frac{\sin(u)}{2} + C = \frac{\sin(x^2)}{2} + C.$$