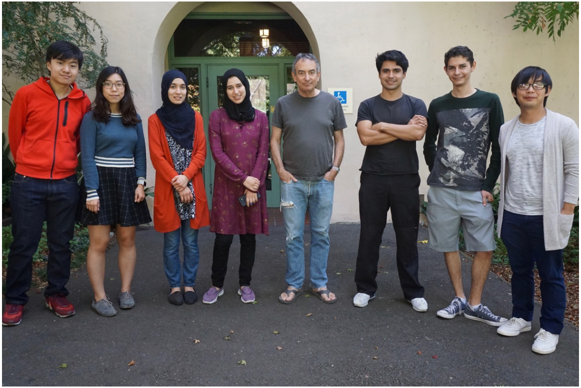
Antiderivatives and area

Math 10A

September 33, 2017

Friday's pop-in lunch:



There is still one slot available for tomorrow's breakfast at 8AM and lots of slots for the new breakfast on Monday, October 9 at 9AM.

The next pop-in Friday lunch will be on October 20 at noon.

Foothill Dinner

Dinner tomorrow (Wednesday, October 4) at 6PM at Foothill DC.

You definitely want to come. See you there!!

Antiderivatives

Last time we started to discuss antiderivatives. For example, the antiderivative of $x^2 + 2$ is $\frac{x^3}{3} + 2x + C$:

$$\int x^2 + 2\,dx = \frac{x^3}{3} + 2x + C.$$

We could also take "initial conditions" into account:

$$F'(x) = x^2 + 2,\ F(0) = 5 \quad \Longrightarrow \quad F(x) = \frac{x^3}{3} + 2x + 5.$$

Connection with area

Imagine now that f is a positive (continuous) function defined on $[a, b]$ (the closed interval of numbers from a to b, with $a \leq b$). We can make sense out of:

"the area under the curve $y = f(x)$ between a and b."

Area is usually defined by approximation: stuff lots of rectangles into the region whose areas you want to calculate, sum up the areas of the rectangles and take a limit as the rectangles become thiner and thiner (and more and more numerous).

Allow me to make a crude sketch on the document camera.

Riemann sums

In approximating areas by sums of areas of rectangles, the approximations that you encounter are called *Riemann sums*. On Thursday, we will see what these sums look like and get experience in recognizing them when they occur "in nature."

Definite integral

The symbol "$\int_a^b f(x)\,dx$" denotes the area under $y = f(x)$ from $x = a$ to $x = b$. The area (written this way) is called the *definite integral* of f between $x = a$ and $x = b$.

Just to repeat: the definite integral is defined as a limit of Riemann sums.

The Fundamental Theorem that we're about to discuss relates the definite integral and the antiderivative $\int f(x)\,dx$. The relation is so tight that the same symbol "$\int$" is used for both.

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Digression

The area of a circle can be defined to be the limit for $n \to \infty$ of the areas of n-sided polygons inscribed in the circle.

Similarly, the perimiter of the circle is the limit of the perimeters of the inscribed polygons.

It's a *theorem* of Archimedes that the perimeter of the unit circle is twice the area of the unit circle. If the area is called π, the perimeter is then 2π.

If you want to see how Archimedes's theorem is proved, we can move to the doc camera. You can also look online.

For the record

We used the definition

$$\pi = \text{area of a circle of radius 1}$$

to show that $\lim_{h\to 0} \dfrac{\sin(h)}{h} = 1$ at the start of the course.

How do we know that the perimeter of this circle is 2π?

Inscribe a regular n-gon in the circle; its perimeter is $2n\sin(\frac{\pi}{n})$.

Using the substitution $h = \frac{\pi}{n}$, $2n = \frac{2\pi}{h}$, we find in the limit

$$\lim_{n\to\infty} 2n\sin(\frac{\pi}{n}) = \lim_{h\to 0} 2\pi\frac{\sin(h)}{h} = 2\pi.$$

Reminder

Areas are defined as limits. We will be explicit about the limits on Thursday.

Area as a new, mysterious function

Think of a as fixed and b as varying. Then the area under $y = f(x)$ from a to b is a function of b:

$A(b)$ = the area under the graph of f between a and b,

$A(x)$ = the area under the graph of f between a and x.

One thing to be said about this function is that

$$A(a) = 0.$$

The area between a and a is 0.

The Fundamental Theorem of Calculus

The fundamental theorem states:

$$A' = f.$$

In words: the derivative of the area function is the function under which you compute the area.

Area is an antiderivative of f.

More precisely, the area function satisfies $A' = f$ and $A(a) = 0$. These two conditions tell you exactly what function area is.

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The log example

Take $f(x) = \dfrac{1}{x}$ and let $a = 1$. Then the area function $A(b)$ represents the area under $y = 1/x$ from 1 to b.

This function satisfies $A'(x) = 1/x$ and $A(1) = 0$.

Therefore, $A = \ln$.

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Why is the fundamental theorem true?

The derivative $A'(b)$ is the limit as $h \to 0$ of the fraction

$$\frac{A(b+h) - A(b)}{h}.$$

The numerator represents the area under $y = f(x)$ between $x = a$ and $x = a + h$. If h is small, this sliver of area is very close to being a rectangle with base h and height $f(b)$. Hence

$$\frac{A(b+h) - A(b)}{h} \approx \frac{f(b) \cdot h}{h} = f(b).$$

As $h \to 0$, the approximation becomes better and better. In the limit, it yields an equality:

$$A'(b) = \lim_{h \to 0} \frac{A(b+h) - A(b)}{h} = f(b).$$

Notation for the Fundamental Theorem

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x);$$

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t);$$

$$\frac{d}{dx}\int_a^x f(x)\,dx = f(x).$$

The third equation is problematic because “x” appears in two different roles, but it’s a pretty standard thing that people write.

This may be confusing, so pay attention and reread

Let's say we want to compute the area under $y = \sin t$ from $t = 0$ to $t = \pi$. Let $A(x) =$ area under sin from 0 to x. Then $A'(x) = \sin x$ by the fundamental theorem.

Hence

$$A(x) = -\cos x + C$$

for some C. but also

$$0 = A(0) = -\cos 0 + C = -1 + C,$$

so $C = +1$. Thus

$$A(x) = -\cos x + 1, \quad A(\pi) = -\cos(\pi) + 1 = 2.$$

The area in question is 2.

The area under $y = \sin t$ from $t = a$ to $t = b$? It's $A(b)$ with

$$A(x) = -\cos x + C.$$

But

$$0 = A(a) = -\cos a + C, \quad C = +\cos a$$

and thus

$$A(x) = -\cos x - (-\cos a).$$

The area under $y = f(t)$ from $t = a$ to $t = b$? If F is an antiderivative of f, the answer is analogously

$$A(x) = F(x) - F(a), \quad A(b) = F(b) - F(a).$$

Notation

$$F(b) - F(a) = F(x)\Big]_a^b .$$

Summary

Suppose that f is positive on the interval $[a, b]$. To compute the area under $y = f(x)$ from $x = a$ to $x = b$:

1. Find an antiderivative F for f.
2. Compute $F(b) - F(a) = F(x)\Big]_a^b$.
3. That's your area.

People say they're "evaluating F between a and b" when they compute $F(b) - F(a)$.

An example

Find the area under the curve $y = e^x$ *between* $x = 0$ *and* $x = \ln 10$.

We take $F(x) = e^x$. The answer is $e^{\ln 10} - e^0 = 9$.

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Another example

Find the area under the curve $y = e^{-x}$ to the right of the y-axis.

This is an advanced or trick question because "$b = \infty$." But we're game, right?

We take $-e^{-x}$ as antiderivative of e^{-x}. The area is

$$\lim_{b\to\infty} -e^{-x}\Big]_0^b = \lim_{b\to\infty}(e^{-0} - e^{-b}) = 1 - 0 = 1.$$

Areas and integrals that have to be computed as limits are called *improper*.

A signed example

Find the area between the x*-axis and the graph of* $y = \sin x$ *between* $x = -\pi$ *and* $x = +\pi$*.*

By symmetry, the area is twice the area between 0 and π, which we already computed. (It's 2.) Hence the answer is "4."

However, $\int_{-\pi}^{0} \sin x \, dx = -2$, $\int_{0}^{\pi} \sin x \, dx = +2$, $\int_{-\pi}^{\pi} \sin x \, dx = 0$.

The principle

If $f < 0$ on $[a, b]$, $\int_a^b f(x)\,dx$ is supposed to be a negative number; it's the "signed area" between $y = f(x)$ and the x-axis on that interval.

The sign occurs because $y = f(x)$ is below the x-axis, rather than being above the axis as it was in our first discussions.