$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3}\, dx = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3} \frac{\sin(x/5)}{x/5}\, dx = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/15)}{x/15} = \frac{467807924713440738696537864469}{935615849440640907310521750000}\, \pi$$

The first seven integrals are $\dfrac{\pi}{2}$ but the eighth is a bit less.

One calculation, then lots of time for questions

Math 10A

AMERICAN MATHEMATICAL SOCIETY

October 24, 2017

Midterm #2

The second midterm will be 48 hours from now, 8:10–9:30AM in 155 Dwinelle.

It is cumulative in principle, but it will stress the second third of the course.

It "covers" everything through last Thursday's course meeting.

You can bring in one two-sided sheet of notes, just like last time.

Please remember to *explain in words* what you are doing.

Questions $\longrightarrow$ `piazza` (if they haven't been addressed during this course meeting).

Ribet travel

I am leaving for a five-day trip on the morning of the exam. Head GSI Teddy Zhu will be in charge.

Because of my absence, I will *not* be holding office hours on Monday, October 30.

I'll be back on Tuesday, October 31 for our class on Probability spaces and random variables.

By the way, last Thursday I talked about slides for this year's course. You might look at the slides for Last year's course and compare them with the slides this year. I would like to think that this year's slides are better than last year's, but last year's might still be useful for additional examples and perspective.

Serge Lang lecture

This is a quick advertisement for This year's Serge Lang Undergraduate Lecture by Keith Devlin of that place. The title is

> *When the precision of mathematics meets the messiness of the world of people.*

The lecture will be given at 4:10PM on Thursday, November 2 in 60 Evans.

Breakfast

Next breakfast: Wednesday, November 1 at 9AM. Please send me email to sign up. (The previously-announced date of October 25 didn't work with the Faculty Club.)

Lunch

Pop-in lunch tomorrow at 12:00:00 at the Faculty Club.

Please come early because I'll need to leave at 12:50.

Dinner

Foothill DC dinner, Friday, Nov. 3, 6:30PM.

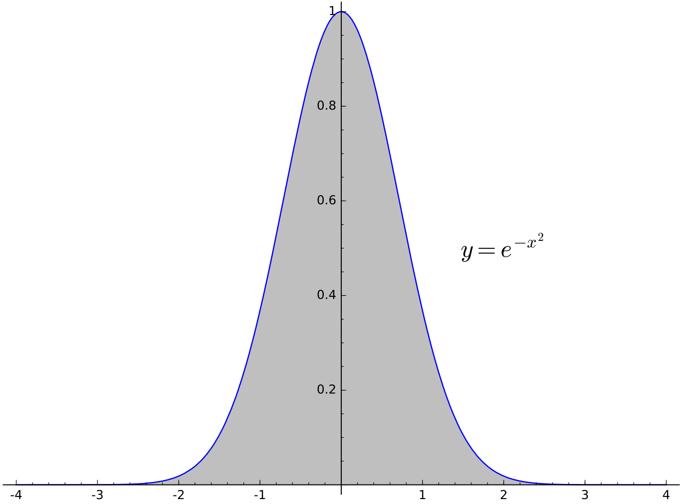
The calculation I'm about to do will not be on the exam.

I want to explain to you a derivation of a fundamental fact that will be used in our discussion of statistics.

Please listen carefully to the argument I'm about to present because it uses some points that are squarely in the scope of the course so far:

- the chain rule;
- the fundamental theorem of calculus;
- the fact that a function is constant if its derivative is 0;
- the calculation $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$, which came up in a class not so long ago.

Consider the bell-shaped exponential curve $y = e^{-x^2}$. The fact to be explained is that the area under this curve is $\sqrt{\pi}$.



In symbols:

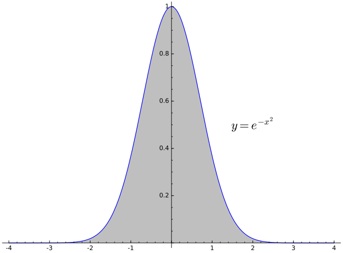
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

$$\updownarrow$$

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi},$$

$$\updownarrow$$

$$\left(\int_{0}^{\infty} e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}.$$



A function

Set

$$F(t) = \left(\int_0^t e^{-x^2}\, dx \right)^2 \text{ for } t \geq 0.$$

Then $F(t)$ is an increasing function of t, and $F(0) = 0$.

The aim is to justify this formula:

$$F(\infty) \stackrel{?}{=} \frac{\pi}{4}.$$

In a previous class, we found

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}.$$

To check that

$$F(\infty) \stackrel{?}{=} \frac{\pi}{4}.$$

it's enough to show that

$$F(\infty) \stackrel{?}{=} \int_0^1 \frac{1}{1+x^2}\,dx.$$

This is basically what we'll do; because F is a function of t, it turns out to be useful to add a parameter "t" to $\int_0^1 \frac{1}{1+x^2}\,dx$.

Adding a parameter

Consider

$$G(t) = \int_0^1 e^{-t^2(1+x^2)} \frac{1}{1+x^2}\, dx.$$

(How do we know that this is intelligent? We are leveraging centuries of experience in the subject!)

Because $\dfrac{1}{1+x^2} \leq 1$ and $e^{-t^2(1+x^2)} \leq e^{-t^2}$, the function of x being integrated is $\leq e^{-t^2}$; therefore

$$G(t) \leq e^{-t^2}.$$

Consequently, $G(t) \to 0$ for $t \to \infty$. Also, $G(0) = \displaystyle\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{4}$.

The trick is to compare F and G by comparing their derivatives. The derivative of $e^{-t^2(1+x^2)}$ with respect to t is

$$e^{-t^2(1+x^2)} \cdot -2t(1+x^2)$$

and therefore

$$\begin{aligned} G'(t) &= \int_0^1 (-2t)e^{-t^2(1+x^2)}(1+x^2)\frac{1}{1+x^2}\,dx \\ &= -2t\int_0^1 e^{-t^2(1+x^2)}\,dx. \\ &= -2te^{-t^2}\int_0^1 e^{-t^2x^2}\,dx. \end{aligned}$$

Meanwhile, we have set

$$F(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2 \text{ for } t \geq 0.$$

The Chain Rule and the Fundamental Theorem of Calculus yield:

$$F'(t) = 2\left(\int_0^t e^{-x^2}\,dx\right)\cdot e^{-t^2}.$$

For comparison:

$$G'(t) = -2te^{-t^2}\int_0^1 e^{-t^2x^2}\,dx.$$

The claim is that $F'(t) = -G'(t)$:

$$\int_0^t e^{-x^2}\,dx = t\int_0^1 e^{-t^2x^2}\,dx.$$

Why does the claim imply $F' = -G'$?

If

$$\int_0^t e^{-x^2}\,dx = t\int_0^1 e^{-t^2x^2}\,dx,$$

then

$$\begin{aligned} G'(t) &= -2te^{-t^2}\int_0^1 e^{-t^2x^2}\,dx \\ &= -2e^{-t^2}\int_0^t e^{-x^2}\,dx = -F'(t). \end{aligned}$$

Verify the claim

$$\int_0^t e^{-x^2}\,dx \stackrel{?}{=} t\int_0^1 e^{-t^2x^2}\,dx.$$

In the integral on the right, put $u = tx$. Then $du = t\,dx$, $dx = \dfrac{1}{t}\,du$; as x runs from 0 to 1, u runs from 0 to t. Hence the right-hand side of the equality to be checked is

$$\int_0^t e^{-x^2}\,dx = \int_0^t e^{-u^2}\,du,$$

which is true because this class is alphabet-neutral.

Summary so far

$$F(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2, \quad F(0) = 0, \quad F(\infty) \stackrel{?}{=} \frac{\pi}{4};$$

$$G(t) = \int_0^1 e^{-t^2(1+x^2)} \frac{1}{1+x^2}\,dx, \quad G(0) = \frac{\pi}{4}, \quad G(\infty) = 0;$$

$$F'(t) = -G'(t).$$

Because $F' + G' = 0$, $F + G$ is some constant.

End of argument

$$F(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2, \quad F(0) = 0, \quad F(\infty) \stackrel{?}{=} \frac{\pi}{4};$$

$$G(t) = \int_0^1 e^{-t^2(1+x^2)} \frac{1}{1+x^2}\,dx, \quad G(0) = \frac{\pi}{4}, \quad G(\infty) = 0;$$

$$F'(t) = -G'(t).$$

Because $F' + G' = 0$, $F + G$ is some constant.

Hence $F + G = \frac{\pi}{4}$ (evaluate both at 0).

Because $G(\infty) = 0$, $F(\infty) = \frac{\pi}{4}$ *as desired*.