Math 10A midterm exam, October 26, 2017

You put away all books, calculators, cell phones and other devices. You consulted a single two-sided sheet of notes. You wrote carefully and clearly, *USING WORDS* (not just symbols). The paper you handed in was your only representative when your work is graded.

The corrected point counts:

Problem	1	2	3	4	5	6	7	Total
Points	5	5	6	6	6	6	6	40

These quick and dirty "solutions" were written by me (Ribet) just before the exam. These are not necessarily perfect, model solutions, but rather my attempt to explain the main ideas.

1. The perimeter of a regular n-gon inscribed in the circle of radius 1 is $2n\sin(\frac{\pi}{n})$. Find the limit as $n \to \infty$ of this expression. (Explain in words what you are doing—this requirement applies to each of the questions on this midterm.)

This was my attempt to placate Archimedes. We calculate

$$\lim_{n\to\infty} 2n\sin(\frac{\pi}{n}) = 2\pi \lim_{n\to\infty} \frac{\sin(\frac{\pi}{n})}{\frac{\pi}{n}} = 2\pi \lim_{h\to 0} \frac{\sin h}{h}.$$

This last limit is known to be 1 (from the beginning of the course), so the answer is 2π.

2a. Show that $\dfrac{1}{m^2+3m+2} = \dfrac{1}{m+1} - \dfrac{1}{m+2}$ for $m \geq 0$.

If we write the difference $\dfrac{1}{m+1} - \dfrac{1}{m+2}$ as a single fraction, the common denominator will be $(m+1)(m+2) = m^2 + 3m + 2$ and the numerator will be $(m+2) - (m+1) = 1$.

b. Find the sum of the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$ by considering the partial sums of the series.

The first terms are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{12}$, so the first three partial sums are $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$. This suggests that the partial sums are of the form $\frac{k+1}{k+2}$ and that the sum of the series will therefore be $\lim_{k\to\infty} \frac{k+1}{k+2} = 1$. Write the nth partial sum as

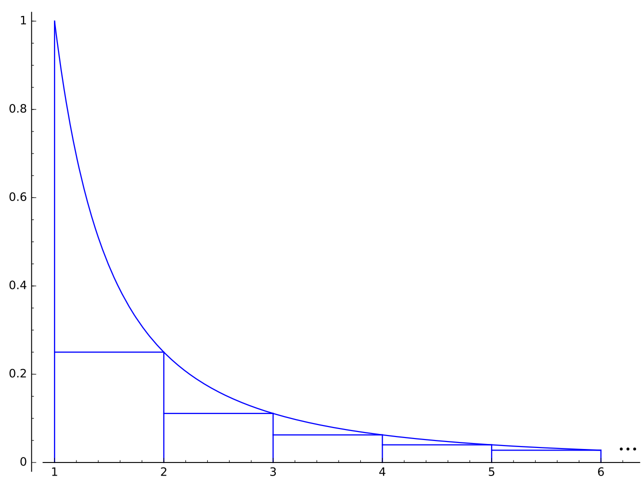
$$(1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots + (\frac{1}{n}-\frac{1}{n+1}) + (\frac{1}{n+1}-\frac{1}{n+2}).$$

There is massive cancellation between adjacent terms, leaving us with the formula that the nth partial sum is $1-\frac{1}{n+2} = \frac{n+1}{n+2}$.

3. Referring to the diagram below, explain carefully why

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le 1 - \frac{1}{n}$$

for $n \ge 2$.



The graph passes through $(1,1)$, $(2,.25)$, $(3,.11)$,.... This suggests that we're looking at the graph of $y = 1/x^2$, and in fact we are. (I know this because I "draw" the graph, using technology.) The picture shows that the rectangle extending from 1 to 2 has height 1/4 and area 1/4 and that it's less than $\int_1^2 \frac{1}{x^2}\,dx$. Similarly, the next rectangle has area 1/9 and is less than $\int_2^3 \frac{1}{x^2}\,dx$. And, as they say, et cetera. The sum of the areas of the rectangles is the left-hand sum in the question and the sum of the integrals is

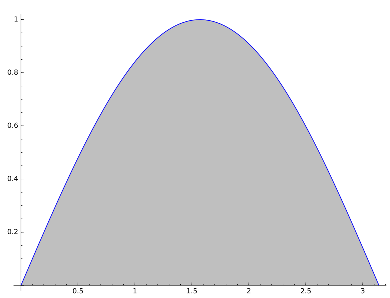
$$\int_1^n \frac{1}{x^2}\,dx = \frac{-1}{x}\Big]_1^n = \frac{1}{1} - \frac{1}{n},$$

which is the right-hand difference in the question. We get the desired inequality because each rectangle is less than a corresponding integral and therefore the sum of the areas of the rectangles is less than the sum of the integrals, which is $1 - \frac{1}{n}$. It follows, by the way, that

$$\sum_{i=2}^{\infty} \frac{1}{i^2} \leq \lim_{n\to\infty} (1 - \frac{1}{n}) = 1.$$

You may know that the infinite sum is $\frac{\pi^2}{6} - 1 \approx 0.64$; this number is indeed less than 1.

4. Determine the volume of the solid obtained by revolving the area under $y = \sin x$ from $x = 0$ to $x = \pi$ about the x-axis. [Hint: it may be helpful to know that $\cos 2x = 1 - 2\sin^2 x$.]



We just have to apply the formula Volume $= \pi \int_a^b y^2\,dx$ with $a = 0$, $b = \pi$, $y = \sin x$. The volume is then

$$\pi \int_0^\pi \sin^2 x\,dx = \frac{\pi}{2} \int_0^\pi (1 - \cos 2x)\,dx = \frac{\pi}{2} \cdot \pi = \frac{\pi^2}{2}.$$

Here I've used the fact that the antiderivative of $\cos 2x$ is $\sin 2x$, up to some factor; the function $\sin 2x$ vanishes at 0 and at π.

5. Find y as a function of x, given $\frac{dy}{dx} = y(2x + 1)$ and $y(0) = 2$.

You acted with honesty, integrity, and respect for others.

This problem is similar to ones you've seen before. Formally, we write

$$\int \frac{1}{y}\,dy = \int (2x+1)\,dx$$

and get $\ln|y| = x^2 + x + C$. Exponentiating gives

$$y = Ke^{x^2+x},$$

where K is some constant. When $x = 0$, $y = 2$. Plugging in this info, we get

$$2 = Ke^0,$$

so $K = 2$. Note that the general solution includes the solution $y = 0$, which we get from $K = 0$. This solution disappears when we do the separation of variables manipulation: we divided by y, assuming implicitly that $y \neq 0$. As I have explained in class (and on `piazza`), the case where K is negative corresponds to the case where $|y| = -y$, and the case where K is positive corresponds to the situation where $|y| = +y$.

6. Use integration by parts, twice, to find an antiderivative of $e^x \sin x$.

This problem was done out in class on October 12. For details, see the notes for that class session. (They're available on bCourses and also from `https://math.berkeley.edu/~ribet/10A/`.)

7. Use the chain rule and the fundamental theorem of calculus to find

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \sin(t^2)\,dt\right).$$

By the fundamental theorem of calculus, the integral may be written as $F(x^3) - F(x^2)$, where F is an antiderivative of the integrand, i.e., where $F'(x) = \sin(x^2)$. By the chain rule,

$$\frac{d}{dx}\big(F(x^3) - F(x^2)\big) = F'(x^3)3x^2 - F'(x^2)2x = 3x^2\sin(x^6) - 2x\sin(x^4).$$

You acted with honesty, integrity, and respect for others.