Math 10A non-final exam, September 26, 2017

You put away all books, calculators, cell phones and other devices. You consulted a single two-sided sheet of notes. You wrote carefully and clearly, *USING WORDS* (not just symbols). The paper you handed in was your only representative when your work is graded.

Point counts:

Problem	1	2	3	4	5	6	7	Total
Points	6	6	6	6	6	5	5	40

These quick and dirty "solutions" were written by me (Ribet) just before the exam. These are not necessarily perfect, model solutions, but rather my attempt to explain the main ideas.

1. Evaluate

$$\lim_{t\to 0} \frac{\sin t - t}{t^3}.$$

The problem is solved at the beginning (after "Announcements") of the slides for the September 19 class meeting. The answer is $\frac{-1}{6}$.

2. Find all points on the interval $[0,1]$ where the instantaneous rate of change of $f(x) = x^3 + x$ is equal to the average rate of change of $f(x)$ on the interval.

The average rate of change in this case is $\frac{f(1)-f(0)}{1-0} = 2$, and $f'(x) = 3x^2+1$. Thus we have to solve the equation $3x^2+1 = 2$, or $x^2 = \frac{1}{3}$. There is only one solution between 0 and 1, namely $\frac{1}{\sqrt{3}}$.

3. One can model the pulse rate (in beats per minute) of a healthy person whose height is x inches by the formula $\frac{592}{\sqrt{x}}$ when $30 \le x \le 100$. Use a linear approximation to estimate the decrease in pulse rate that corresponds to a height increase from 64 to 64.5 inches.

This annoying question was adapted from problem 32 on page 250 of the textbook. If $P(x) = \frac{592}{\sqrt{x}}$, we're trying to estimate $P(64.5) - P(64)$, which is

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numerically around -0.287. The technique of linear approximation gives the estimate $P'(64)(64.5-64) = P'(64)/2$ for the negative number $P(64.5) - P(64)$. We can say that the *decrease* of P between the two values is approximately $-P'(64)/2$. Now $P'(x) = -592 \cdot \dfrac{1}{2}x^{-3/2} = -296x^{-3/2}$ and thus $-P'(64) = 296/64^{3/2} = \dfrac{296}{64 \cdot 8} = 37/64$. Finally, $-P'(64)/2 = 37/128$, which is the answer if I didn't make any arithmetic errors. Note that $37/12 \approx 0.289$, so it looks as if I did the calculation correctly.

(As I've explained, you can leave your answer severely unsimplified. It's in your interest to do that because you'll lose points if you simplify and mess up.)

4. Calculate

$$\lim_{t \to -\infty} \sqrt{t^2+t}+t.$$

If you set $x = -t$, the limit to be calculated becomes

$$\lim_{x \to \infty} \left(\sqrt{x^2-x}-x\right)$$

in terms of x. This is very much like several other "$\infty - \infty$" limits we've seen in the course. Write $\sqrt{x^2-x}-x$ as

$$\frac{(x^2-x)-x^2}{\sqrt{x^2-x}+x} = \frac{-x}{\sqrt{x^2-x}+x}$$

and divide numerator and denominator by x:

$$\frac{-1}{\sqrt{1-1/x}+1} \longrightarrow \frac{-1}{1+1} = -1/2.$$

5. Calculate $\dfrac{d}{dx}\left(\sqrt{\sin(x^2)}\right)$.

You have to use the chain rule twice:

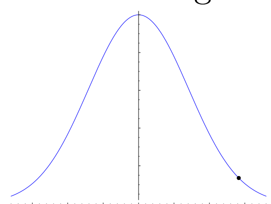
$$\frac{d}{dx}\left(\sqrt{\sin(x^2)}\right) = \frac{1}{2}\frac{1}{\sqrt{\sin(x^2)}} \cdot \frac{d}{dx}\left(\sin(x^2)\right)$$

and

$$\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2)2x.$$

Thus the final answer is very likely $\dfrac{x\cos(x^2)}{\sqrt{\sin(x^2)}}$.

6. Find the coordinates of the inflection point on the graph of $y = e^{-x^2}$ that lies to the right of the y-axis.



Let $f(x) = e^{-x^2}$. You find the inflection point a by setting $f''(a) = 0$. The second derivative (calculated using the chain rule and the formula for the derivative of a product) is $(4x^2 - 2)e^{-x^2}$, if I'm not mistaken. This expression is 0 when $x = \pm\dfrac{1}{\sqrt{2}}$. Since we're looking to the right of the y-axis, the x-coordinate is positive. The point occurs for $x = \dfrac{1}{\sqrt{2}}$ and the coordinates of the point are $(\dfrac{1}{\sqrt{2}}, e^{-1/2})$.

7. Let $a_n = \dfrac{n^n}{n!}$. Decide whether the limit $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists and evaluate the limit if it does. [Note that $n! = 1\cdot 2\cdot 3\cdots(n-1)\cdot n$ and that our textbook defines e as the limit $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.]

See the slides of the September 21 class meeting for some relevant discussion.

We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = (n+1)\frac{(n+1)^n}{n^n}\frac{n!}{(n+1)!}.$$

Now $(n+1)n! = (n+1)!$, so

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n \to e.$$

The limit does exist; its value is e.

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