Math 10A final exam, December 14, 2017

This was a 3-hour exam. Students had access to a one-page two-sided "cheat sheet" but were not allowed to bring in devices of any kind (e.g., phones, laptops, calculators, modern watches) or to have other written material.

The point counts for the nine problems were as follows:

Problem	1	2	3	4	5	6	7	8	9	Total
Points	8	9	7	9	9	6	8	6	8	70

These solutions were written up very quickly by me, Ribet. If you notice errors or imprecisions, please let me know and I'll update the document. I'm pretty sure that these questions, along with their skeletal solutions, will be of great interest to students who take Math 10A in the future. Let's try to make sure that we don't confuse them unnecessarily.

1. Compute:

a. $\lim\limits_{x\to\infty} \big(x - \ln(1+e^x)\big)$,

This is a classic tug of war problem that looks formally like the indeterminate expression "$\infty - \infty$." The limit is the natural log of the limit

$$\lim_{x\to\infty} \frac{e^x}{1+e^x} = \lim_{x\to\infty} \frac{1}{e^{-x}+1} = \frac{1}{1} = 1.$$

Hence the limit is 0. You can compute $\lim\limits_{x\to\infty} \dfrac{e^x}{1+e^x}$ by l'Hôpital's Rule, and I suspect that most people will do it that way.

b. $\lim\limits_{t\to\infty} \dfrac{t}{\sqrt{t^2-1}}$.

I think that you get into an infinite loop if you try to use l'Hôpital's Rule on this problem, compare problem #2 on page 273 of the textbook. But you can divide numerator and denominator by t and write the limit as $\lim\limits_{t\to\infty} \dfrac{1}{\sqrt{1-t^{-2}}}$. Then it's clear from this form of the expression that the limit will be $\dfrac{1}{1} = 1$.

2. Explain carefully why the function

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ \dfrac{1}{x^2} & \text{if } x \geq 1 \end{cases}$$

is a probability density function. Compute the corresponding cumulative distribution function (CDF) and sketch a graph of the CDF.

I don't know why I used the word "carefully"—the idea is surely to explain everything carefully! I also don't know how we managed to misspell "probability" on the exam sheet! For f to be a PDF, we need it to be non-negative (which it clearly is) and need $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$. By definition, the integral in question is $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$. An antiderivative of the integrand is $\dfrac{-1}{x}$, which is -1 at 1 and 0 at ∞. Hence the integral is 1, as required.

The CDF is 0 for $x \leq 1$ and is $1 - \dfrac{1}{x}$ for $x \geq 1$. I won't try to sketch that function in this write-up.

3. Which is more likely: getting 60 or more heads when a fair coin is tossed 100 times, or getting 175 or fewer heads when a fair coin is tossed 400 times?

We associate z-scores to both situations. For the first scenario, the z-score is positive. For the second, it's negative. The larger the z-score in absolute value, the smaller the associated probability. Hence the more likely scenario is the one with the smaller z-score (in absolute value). In the two cases, the z-scores are given by the formula $(\overline{x} - \mu) \cdot \dfrac{\sqrt{n}}{\sigma}$ where $\mu = 0.5$ and $\sigma = 0.5$ as well. We can ignore the factor σ because it's the same for the two scenarios. In the first scenario $(\overline{x} - 0.5)\sqrt{n} = 0.1 \times 10 = 1$. In the second scenario, $|(\overline{x} - 0.5)\sqrt{n}| = \dfrac{25}{400} \times 20 = \dfrac{25}{20} > 1$. Hence the second scenario is more unlikely. The more likely scenario is the first one: getting 60 or more heads when a fair coin is tossed 100 times.

4. For $z \geq 0$, let $T(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^z e^{-x^2/2}\,dx$. (Values of this frequently tabulated function appear on page 566 of our textbook.) Write in terms of T:

a. $P(X \geq 0)$ when X is normally distributed with mean -1 and standard deviation 2.

This problem is like the first Drill Problems at the end of §7.4 of the book. The general formula for the PDF of a normal variable is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

so the probability to be computed is

$$\frac{1}{2\sqrt{2\pi}} \int_0^\infty e^{-\frac{(x+1)^2}{8}}\, dx = \frac{1}{2\sqrt{2\pi}} \int_1^\infty e^{-t^2/8}\, dt.$$

We have made the change of variable $t = x+1$. Now let $u = t/2$, $t = 2u$, $dt = 2\,du$. The integral becomes

$$\frac{1}{\sqrt{2\pi}} \int_{1/2}^\infty e^{-u^2/2}\, du = T(\infty) - T(1/2).$$

Since $T(\infty) = \dfrac{1}{2}$, the answer is $\dfrac{1}{2} - T(1/2)$. Looking at the actual printed table on page 566, I see the value 0.1915 for $T(1/2)$. Thus the answer is numerically around 0.31.

b. $P(-2 \leq X \leq 0.5)$ when X is normally distributed with mean 0 and standard deviation 1.

The variable X is a standard normal variable (mean 0, standard deviation 1). Therefore, the probability in question is the area under the standard normal curve between -2 and 0.5. That is the sum of the area from -2 to 0 and the area from 0 to 0.5. The first area is the same as the area from 0 to 2. Accordingly, the answer is $T(0.5) + T(2)$.

5a. Find the number C so that

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ Ce^{-x^2/2} & \text{if } x \geq 0 \end{cases}$$

is a probability density function.

The integral of $e^{-x^2/2}$ over $(-\infty, \infty)$ is $\sqrt{2\pi}$; you're sort of reminded of this well known fact in the statement of problem 4. Our function (without the C) is 0 to the left of the y-axis and $e^{-x^2/2}$ to the right of the y-axis. Hence its integral over $(-\infty, \infty)$ is $\frac{1}{2}\sqrt{2\pi}$. Therefore, C needs to be $\frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$.

b. If X is a random variable whose PDF is $f(x)$, what is the expected value of X? (You can write your answer in terms of C or plug in the value of C that you found in part a.)

The answer is $C\int_0^\infty xe^{-x^2/2}\,dx = -C\,e^{-x^2/2}\Big]_0^\infty = C$. (I did this quickly and hope I didn't mess up.)

6. Find all numbers a and b such that

$$y(t) = ae^{bt}$$

is a solution to the differential equation

$$y''(t) = 3y'(t) - 2y(t).$$

The derivative of ae^{bt} is abe^{bt}, and the second derivative of ae^{bt} is ab^2e^{bt}. The equation $y''(t) = 3y'(t) - 2y(t)$ becomes $e^{bt} \cdot a \cdot (b^2 - 3b + 2) = 0$. The function e^{bt} is not 0 for any value of t, so we have the identity of functions $y''(t) = 3y'(t) - 2y(t)$ if and only if $a(b^2 - 3b + 2) = 0$. The equation is satisfied if $a = 0$, in which case there is no restriction on b. If a is non-zero, the equation $a(b^2 - 3b + 2) = 0$ holds only if $b^2 - 3b + 2 = 0$, i.e., exactly when $b = 1$ or $b = 2$. Hence the differential equation is satisfied for all b when $a = 0$ and for all a when $b = 1, 2$.

7a. Find $\frac{dy}{dx}$ if $x = \frac{y}{1-y^2}$. (Your answer may involve both x and y.)

Differentiate both side of the equation $x = \frac{y}{1-y^2}$ with respect to x:

$$1 = \frac{d}{dy}\left(\frac{y}{1-y^2}\right) \cdot \frac{dy}{dx} = \frac{(1-y^2) - y(-2y)}{(1-y^2)^2} \cdot \frac{dy}{dx} = \frac{1+y^2}{(1-y^2)^2} \cdot \frac{dy}{dx}.$$

We get

$$\frac{dy}{dx} = \frac{(1-y^2)^2}{1+y^2}.$$

b. Evaluate the indefinite integral $\displaystyle\int x^2 \sin x\, dx$.

Integrate by parts, twice. On the first round, we put $u = x^2$, $dv = \sin x\, dx$, $v = -\cos x$ and get

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx.$$

Similarly, $\displaystyle\int x\cos x\, dx = x\sin x + \cos x + C$. All together, we end up with

$$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

8. Let $f(x) = x^3 - 16x$, $a = -4$, $b = 2$. Find all numbers c with $a < c < b$ such that the line tangent to $y = f(x)$ at $(c, f(c))$ is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$.

I get $f(a) = 0$, $f(b) = -24$. The slope of the secant line is $\dfrac{-24}{6} = -4$. Also, $f'(x) = 3x^2 - 16$, so we want to solve

$$3c^2 - 16 \stackrel{?}{=} -4.$$

We get $3c^2 = 12$, $c = \pm 2$. Because we need to have $-4 < c < 2$, it looks as if $c = -2$ is the only answer here.

9a. Find a positive number C such that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{a^n n!}$ converges for $a > C$ and diverges for $0 < a < C$.

Let $b_n = \dfrac{n^n}{a^n n!}$, so the series is $\displaystyle\sum_{n=1}^{\infty} b_n$. We can try the ratio test to see whether it sheds light on convergence. We compute

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^{n+1}}{a^{n+1}(n+1)!} \cdot \frac{a^n n!}{n^n} = \left(\frac{n+1}{n}\right)^n \frac{1}{a} \longrightarrow \frac{e}{a}.$$

By the ratio test, the series converges for $\frac{e}{a} < 1$ and diverges for $\frac{e}{a} > 1$. In other words, there is convergence for $a > e$ and divergence for $a < e$. We have found the required value of C: it's e.

b. Decide whether or not this series converges:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} \cdots.$$

(Each term $\frac{1}{n}$ occurs n times.)

A series of positive numbers diverges if and only if the partial sums of the series march off to infinity. Among the partial sums of this series are all of the positive integers. For example, $\frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4$. As you add more and more terms, you go beyond 1, 2, 3, 4, 5, and so on. The series diverges to ∞.