Welcome to Math 10A!!

Kenneth A. Ribet

August 29, 2017

An experiment?

I started writing these slides with the idea that they would be deposited on `bCourses` before our class meeting and that I'd avoid projecting them. My plan was to use the chalkboards and possibly the document camera.

Now I like them enough that I want to project them.

Your job is to get me to use the chalkboard and document camera and thereby take me away from mindless recitation of what's on the screen in front of you.

Needless to say, I hope you'll give me lots of feedback on the usefullness of slides.

In any case, I'd like you to look at all of the slides in this file, even if some of the examples in the slides don't get discussed in class.

More books

There are a handful of additional 2016 Math 10A students who want to sell their textbooks:

Lillian Tran (maybe already sold)
Raquel Camacho
Diana Sosa
Haseena Momand
Mersal Danai

(Click on a name to send email to the seller.) You can pick up these slides on `bCourses`.

Social events

There was a breakfast on Monday morning...

and a dinner at Crossroads on Monday evening:

These upcoming breakfasts are full: August 30, September 11, September 18.

There's a new breakfast at 8AM on Wednesday, September 20. Please send me email if you'd like to come.

The next pop-in lunch will be on Friday, September 1 at 11:45AM at the Faculty Club.

Where are we in the book?

I'd like *not* to talk today about exponential and logarithmic functions.

Instead, I'd like to start with *sequences*. A sequence is just a function on a set of integers. There are a couple of such functions at the end of the notes from last Thursday. For example, there's the function $F(n) :=$ the number of ways of dividing a class with n students into two-person study groups.

Sequences are most frequently written a_n or (a_n) instead of $F(n)$.

Wikipedia's treatment of sequences is amusing and enlightening. One point is that Wikipedia's sequences don't have to have numbers as values; the sequence a_n where a_n is the *n*th room you've entered since arriving at UC Berkeley is a perfectly good sequence for Wikipedia.

For this course, functions are numerical, so it might be better to let a_n be the *volume* of the *n*th room you've entered since arriving at UC Berkeley.

More precisely, we'd need to say "volume *in cubic meters*" in order to get a number.

Sequences can be described by a rule, for example:

$$p_n = \text{the } n\text{th prime number}, \quad p_1 = 2, p_2 = 3, \ldots.$$

Sequences can be described by a formula:

$$a_n = \sin(n \cdot \frac{\pi}{2}) \text{ for } n = 0, 1, 2, \ldots.$$

Note $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -1$, $a_4 = 0$, etc., etc. The sequence is periodic with period 4.

The informal description

"$a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -1$, $a_4 = 0$, $a_5 = 1$, etc."

is probably solid enough that most mathematicians would understand what is intended and wouldn't object to it.

Limits of sequences

Probably the most important theme in this (tiny) part of the course is that sequences can "approach" a limit, meaning that a_n gets closer and closer to some number as n approaches infinity. Example:

$$a_n = \frac{1}{n} \text{ for } n \geq 1$$

approaches 0 as $n \to \infty$. As the denominator gets bigger and bigger (and the numerator sits as 1), the fraction approaches 0.

My idea now is to introduce a bunch of noteworthy sequences.

Why am I do this? To try to convince you that sequences can be interesting. They can approach somewhat surprising limits. There are lots of different types of examples.

In cases where a sequence approaches a limit ("Fact"), I'll try to explain in a few words what's going on.

Let $f_n = n\sin(\frac{1}{n})$ for $n = 1, 2, 3, \ldots$. What is the behavior of f_n for n big?

Some data:

$$f_{10} = 0.84, f_{10^2} = 0.96, f_{10^3} = 0.98, f_{10^4} = 0.989, f_{10^5} = 0.993.$$

Fact: $f_n \to 1$ as $n \to \infty$.

This "Fact" results from the statement that the derivative of sin is cos. We don't yet know what derivatives are; even if we knew the definition of a derivative, we'd have to have a further discssion to see how to differentiate trigonometric functions.

BTW, many, many Math 10A students have never seen calculus.

A second example, involving a sum

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}, \quad n \geq 1.$$

As $n \to \infty$, the sum s_n approaches $\frac{\pi^2}{6}$. This is very far from obvious but can be proved using sophomore calculus and isn't on the syllabus of Math 10 (as far as I know).

Terminology: we're saying that the infinite sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

has a meaning and that its value is $\frac{\pi^2}{6}$. One says that the series *converges* to $\frac{\pi^2}{6}$.

The analogous sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

doesn't have a numerical value—one says that it *diverges* to infinity.

This follows from the *integral test*, which we will discuss in mid-September. The Schreiber book does not have a lot of material about infinite series, but there's a discussion in the .pdf textbook for the course.

Let

$$c_n = \left(1 + \frac{1}{n}\right)^n, \quad n \geq 1.$$

The sequence c_n begins $c_1 = 2$, $c_2 = 9/4$, $c_3 = 64/27, \ldots$.

Numerically, the first 10 c_n are:

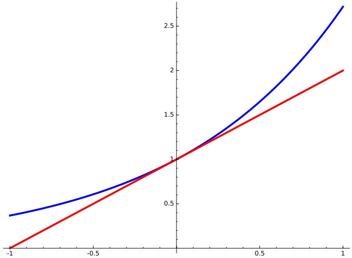
$$2, 2.25, 2.37, 2.44, 2.488, 2.5216, 2.546, 2.565784, 2.58.$$

Further, $c_{1000} \approx 2.7169$, $c_{10000} \approx 2.71814592682522$, $c_{100000} \approx 2.71826823717449$.

Fact: $c_n \to e$, where $e \approx 2.71828182845905$.

Why is this true? In our book (p. 54), it's the very *definition* of e. You have to believe that the sequence converges to *something*, and then you can say that e is the limit.

Another definition of e



The (blue) graph of $y = e^x$ kisses the (red) line $y = x + 1$ at the point $(0, 1)$ where it crosses the y-axis. The appeal of the line in question is that it has slope equal to 1.

In other words, the curve $y = e^x$ has *slope* equal to 1 at the point $(0, 1)$. If we took $y = a^x$ with $a > 0$, $a \neq e$, the slope would be a number other than 1; in fact, the slope would be $\ln a$.

Whoops—we're doing calculus now. Stay tuned for more.

You've probably heard of the Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,$$

which appear in Math 10B. The rule is that each number is the sum of the preceding two.

Consider the sequence of ratios of successive Fibonacci numbers:

$$1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13.$$

Numerically:

$$1, 2, 1.5, 1.66, 1.6, 1.625, 1.615384.$$

These ratios go alternately up and down, as you can see from the examples. (2 is bigger than 1, 1.5 is less than 2, 1.66 is bigger than 1.5, etc.)

Fact: The ratios approach $\dfrac{1+\sqrt{5}}{2} \approx 1.6180$.

This fact can be deduced pretty easily from an explicit formula for the nth Fibonacci number that you're likely to see in Math 10B.