Derivatives and limits

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Announcements

The breakfasts on September 7 and September 8 are full, but there are spaces at the 8AM breakfast on September 13.

Homework assignments will be available from the “homework” section of the class web page

https://math.berkeley.edu/~ribet/10A/.

Chapter 2 of the Schreiber book is available in .pdf form from bCourses.

I'm going to begin with some limit problems that involve sequences. I'll put the limit on the screen, and we can discuss it together—possibly using the chalkboards or the document camera.

Ready?

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = ?$$

The answer to the first question is that the limit is zero. To really see why that's so, we have to know what it means to approach something! For the matter at hand, suppose we have a sequence (a_n) and want to understand the statement that $a_n \to 0$ as $n \to \infty$.

The definition is that if you have a tiny number, say 10^{-9}, then the absolute value of a_n is less than that small number for all n "sufficiently large." In the case of the first limit, this means that, for n beyond some point, $(\frac{1}{2})^n$ is less than 10^{-9}.

It's pretty clear that this is so because $2^n > 10^9$ for all sufficiently large n—in fact for $n \geq 30$. To make a "proof," we have to do this with 9 replaced by an arbitrary number, but we'd be OK doing that.

To say that $\lim_{n\to\infty} a_n = L$ is to say something precise about $|a_n - L|$ for n big. The exact formulation can be phrased this way:

If you take a negative power of 10, say $\dfrac{1}{10^k}$, then we'll have $|a_n - L| < \dfrac{1}{10^k}$ for all sufficiently large n.

This means: for each $k \geq 1$, there's an $N \geq 1$ (depending on k) such that $|a_n - L| < \dfrac{1}{10^k}$ for all $n \geq N$.

$$\lim_{n\to\infty} \alpha^n = ?$$

Here, α is understood to be a real number. If we can figure out what happens for $\alpha = -7$, $\alpha = -1$, $\alpha = -\frac{1}{3}$, $\alpha = 0$, $\alpha = \frac{7}{8}$, $\alpha = +1$, $\alpha = 1.1$, then we're in really good shape.

$$\lim_{n\to\infty}(1+\alpha+\alpha^2+\cdots+\alpha^{n-1}) = ?$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = ?$$

$$\lim_{n\to\infty}\left(\frac{n^2+3n+1}{2n^2-n+1}\right)=?$$

$$\lim_{n\to\infty} \frac{n^3 + 2n}{n^4} = ? \qquad \lim_{n\to\infty} \frac{n^2 + 2n}{n + 1} = ?$$

$$\lim_{n\to\infty} \frac{n^2 - n}{n} = ?$$

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = ?$$

One way to think about the previous problem is to look at

$$\lim_{a \to \infty} \left(\sqrt{a+1} - \sqrt{a} \right),$$

i.e., to avoid thinking that the argument is an integer. (It might as well be a random positive real number.)

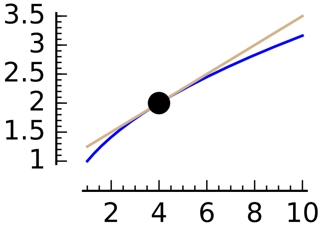
How much does the square root of a differ from the square root of $a+1$?

This is a question about the average rate of change of the square root function over the interval $[a, a+1]$:

$$\frac{\sqrt{a+1} - \sqrt{a}}{(a+1) - a}.$$

This average rate is plausibly not so different from the *instantaneous* rate of chage of the square root function at a, i.e., the derivative of the square root function at a.

This derivative is the slope of the line tangent to the curve $y = \sqrt{x}$ at the point $(a, \sqrt{a})$.



The picture suggests that $\sqrt{a+1} - \sqrt{a} \leq m$, where m is the slope of the tangent line. (This is true.)

One can compute the slope of the tangent line in two ways. The first is to view it as the derivative $f'(a)$, where $f(x) = \sqrt{x}$:

$$m = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}.$$

The fraction is

$$\frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}} = \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})}$$

or $\dfrac{1}{\sqrt{a+h} + \sqrt{a}}$, which approaches $\dfrac{1}{2\sqrt{a}}$ as $h \to 0$. In other words, the slope is $\dfrac{1}{2\sqrt{a}}$.

The second way to figure out the slope is to say that the graph of $y = \sqrt{x}$ is the graph of $x = y^2$. It's thus the graph of $y = x^2$ with the roles of x and y reversed. The point $(a, \sqrt{a})$ on the square root graph corresponds to the point $(\sqrt{a}, a)$ on the squaring graph.

Now on the squaring graph, we figured out on Tuesday that the slope of the line tangent to a point (b, b^2) is $2b$. Thus, on the squaring graph, the slope of the line tangent to the point $(\sqrt{a}, a)$ is $2\sqrt{a}$. However, when you now reverse the roles of x and y, the slope becomes the reciprocal $\frac{1}{2\sqrt{a}}$. That's because a slope is "change in y divided by change in x."

It's validating that we get the same answer from the two different perspectives.

If you believe the picture (which I do), then you see that

$$\sqrt{a+1} - \sqrt{a} \leq \frac{1}{2\sqrt{a}}$$

for $a > 0$. As $a \to \infty$, the quantity $\dfrac{1}{2\sqrt{a}}$ approaches 0, and so the difference must approach 0 as well.

Technically, we're using the ("squeeze") principle that a non-negative quantity approaches 0 if it is less than something that's approaching 0.

In fact, we didn't do this limit problem in the most direct way. Go back to the formula

$$\frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \text{(something equal to 1)} = \frac{(a+h)-a}{h(\sqrt{a+h}+\sqrt{a})}$$

and take $h = 1$. You get

$$\sqrt{a+1}-\sqrt{a} = \frac{1}{\sqrt{a+1}+\sqrt{a}}.$$

When $a \to \infty$, the terms in the denominator both approach infinity and the fraction approaches 0.