Substitution and change of variables
Integration by parts

Kenneth A. Ribet

Math 10A
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Announcements

I have been back since Friday night but will be leaving for another short trip on Thursday. James will preside over Thursday's class. I will see you again next week for office hours M Tu Th and for our two classes.

The next breakfast will be on Thursday, October 13 at 9AM—two days from now. There are still plenty of open slots. If you'd like to come, please send me email. (I won't be leaving town until after the breakfast!)

The next pop-in lunch will be on Wednesday, October 19 at 12:15PM.

See `piazza` for a discussion of a lunch at VIK's Chaat on Thursday, October 20.

Where we're at

I hope that we are now all increasingly comfortable with these points:

- The integral $\int_a^b f(x)\,dx$ is defined as a limit of Riemann sums.
- This integral measures area.
- The derivative of $\int_a^x f(t)\,dt$ (with respect to x) is $f(x)$.
- If $F(x)$ is a primitive (a. k. a. antiderivative, a. k. a. indefinite integral) of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b.$$

Where we're going

The aim of the next discussion(s) is to explore techniques for finding $F(x)$ when $f(x)$ is given to us. Usually we are presented with a problem that's phrased like this:

$$\text{Find } \int \sin^3 x \, dx.$$

This means that we're supposed to exhibit a function whose derivative is the cube of $\sin x$.

When I was a student, I learned a plethora of techniques for solving problems like this by reading my calculus textbook. Nowadays, it is perhaps more appropriate to turn to YouTube.

Alternatively, we can use software like Wolfram Alpha or `Sage`.

Stephen Wolfram

When I asked `Sage` to integrate $\sin^3 x$, I immediately got the answer

$$\frac{1}{3}\cos(x)^3 - \cos(x),$$

which I hope is right!

There are certainly alternative forms of the answer. For example, Wolfram gives

$$\frac{1}{12}(\cos(3x) - 9\cos(x)) + C.$$

When you have an answer—your answer—and an allegedly correct answer, it's sometimes a challenge to check that they agree up to a constant.

For the rest of the class period, I will propose integration problems and try to do them out in front of you, thereby introducing you to the techniques of *integration by substitution* and *integration by parts*.

Calculate the integral

$$\int \sin(2x)\, dx.$$

First solution

Let $u = 2x$. Then $\dfrac{du}{dx} = 2$, so $du = 2dx$ and $dx = \dfrac{1}{2}du$. The integral is

$$\int \sin u \, \frac{1}{2} du = \frac{1}{2} \int \sin u \, du = -(\cos u)/2 + C = -\cos(2x)/2 + C.$$

Second solution

Remember that $\sin(2x) = 2\sin x \cos$, so the integral to be calculated is

$$2\int \sin x \cos x \, dx.$$

Put $u = \sin x$, so $du = \cos x \, dx$ and the integral becomes

$$2\int u \, du = u^2 + C = (\sin x)^2 + C.$$

Third solution

Because $\sin(2x) = 2\sin x \cos$, the integral to be calculated is

$$2\int \sin x \cos x \, dx.$$

Put $u = \cos x$, so $du = -\sin x \, dx$ and the integral becomes

$$-2\int u \, du = -u^2 + C = -(\cos x)^2 + C.$$

Comparison: are the second and third solutions the same? Do $(\sin x)^2$ and $-(\cos x)^2$ differ by a constant? Yes: their difference is $\sin^2 x + \cos^2 x = 1$.

Further comparison: does $-\cos(2x)/2$ differ from $-\cos^2 x$ by a constant? Remember that $\cos(2x) = \cos^2 x - \sin^2 x$, so

$$-\cos(2x)/2 - (-\cos^2 x) = \frac{1}{2}(\sin^2 x - \cos^2 x) + \cos^2 x = \frac{1}{2}.$$

A definite integral

Find the area under $y = \dfrac{x}{1+x^2}$ between the y-axis and the line $x = 1$.

This area is $\displaystyle\int_0^1 \frac{x\,dx}{1+x^2}$.

Put $u = 1 + x^2$, $du = 2x\,dx$. As x runs from 0 to 1, u runs from 1 to 2. The integral becomes

$$\frac{1}{2}\int_1^2 \frac{du}{u} = \frac{1}{2}\ln u\,\Big]_1^2.$$

The answer is thus $\dfrac{1}{2}\ln 2 = \ln(\sqrt{2})$.

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Evaluate $\int_0^1 \frac{dx}{1+x^2}$.

It's possible to do this quickly if you remember that the derivative of the arctan function is $\frac{1}{1+x^2}$. That's kinda unlikely. Better is to remember that $1+x^2$ has something to do with triginometry; there's an identity

$$1+\tan^2 = \sec^2 .$$

The key is then to put $x = \tan\theta$. Note that previously we introduced the new variable (u) as a function of x. Now we are going backwards and introducing θ by writing x as a function of θ. (Of course, $\theta = \arctan x$.)

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Now $x = \tan\theta$, so

$$\frac{dx}{d\theta} = \sec^2\theta = 1 + \tan^2\theta = 1 + x^2.$$

Moreover, as x runs from 0 to 1, θ runs from 0 to $\pi/4$. We get

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^{\pi/4} \frac{(1+x^2)\,d\theta}{1+x^2} = \int_0^{\pi/4} 1\,d\theta = \frac{\pi}{4}.$$

Variant: Find

$$\int_0^\infty \frac{dx}{1+x^2}.$$

(Note that ∞ means $+\infty$.)

The key is to proceed as before. Technically, by the way,

$$\int_0^\infty \frac{dx}{1+x^2} := \lim_{T\to\infty} \int_0^T \frac{dx}{1+x^2},$$

but we will just be fast and loose with the limits. The idea is that θ runs from 0 to $\pi/2$ as x runs from 0 to ∞, so the integral to be computed is now

$$\int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}.$$

Compute

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

We are computing an area that's twice the previous area, so the answer is now $2 \cdot \frac{\pi}{2} = \pi$.

Evaluate the integral

$$\int_0^\infty xe^{-x^2/2}\,dx.$$

If $u = x^2/2$, then $du = x\,dx$, so the integral is

$$\int_0^\infty e^{-u}\,du = -e^{-u}\Big]_0^\infty = 1.$$

Integration by parts

The formula for the derivative of a product of two functions of x may be written

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Formally, we can multiply by dx to get the amazing formula

$$d(uv) = u\,dv + v\,du.$$

We can then integrate to write

$$\int u\,dv = uv - \int v\,du.$$

What this means is that, faced with integrating an expression like $f(x)g(x)\,dx$, we can reduce the integral that we want to evaluate with a second integral in which f has been replaced by its derivative and g by one of its antiderivatives.

Example

Calculate $\int x \sin x \, dx$.

We let $u = x$ and $\sin x \, dx = dv$. Then we'll take v to be $-\cos x$ and we get

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C.$$

Check: the derivative of $\sin x$ is $\cos x$ and the derivative of $-x \cos x$ is $-\cos x + x \sin x$. The sum of the two terms is $x \sin x$, as desired.

Example

Find $\int \ln x \, dx$.

The key is to take $v = x$ and $u = \ln x$. Then

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - x + C.$$

Example

Calculate $\int_0^\infty xe^{-x}\,dx$.

We take $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$. Then

$$\int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\,dx.$$

The first term is 0 and the second term is evaluated to be

$$-e^{-x}\Big]_0^\infty = 1.$$