Math 10A midterm exam, September 22, 2016

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly, *USING WORDS* (not just symbols). Remember that the paper you hand in will be your only representative when your work is graded.

What follows is a discussion of the midterm problems that was written by me (Ribet) just before the exam. These are not necessarily perfect, model solutions, but rather my attempt to explain what's going on.

1. Find a formula for the derivative $\dfrac{dy}{dx}$ in each of the following three cases:

$$\textbf{a. } y = \sqrt[3]{\sin(x^3)}, \quad \textbf{b. } y = \ln\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right), \quad \textbf{c. } x^4 + 4y^4 = 7.$$

In the third case, your formula may involve y as well as x.

For (a): By the chain rule, the derivative is $\frac{1}{3}(\sin(x^3)^{-2/3})$ times the derivative of $\sin(x^3)$. Again by the chain rule, this latter derivative is $\cos(x^3)$ times the derivative of x^3, which is $3x^2$. The answer is thus a product of three terms. I asked my computer for the derivative (using `Sage`) and got back

$$\frac{x^2\cos(x^3)}{\sin(x^3)^{2/3}}$$

as the answer. Note that the $1/3$ at the beginning of the problem is cancelled by the 3 at the end of the problem.

In (b), I will write $\sec x$ for $\dfrac{1}{\cos x}$ and $\tan x$ for $\dfrac{\sin x}{\cos x}$. This is not necessary to do the problem, but I'm hoping that the solution will involve less typing for me if I go that way. The derivatives of $\sec x$ and $\tan x$ are respectively $(\sec x)(\tan x)$ and $\sec^2 x$, as you can check from the quotient rule. The derivative of $\ln(\sec x + \tan x)$ is then

$$\frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x) = \frac{\sec^2 x + (\sec x)(\tan x)}{\sec x + \tan x}.$$

In this fraction, each term in the numerator is $\sec x$ times the term in the denominator that's below it. Hence the final answer is $\sec x$. (The takeaway is that if someone asks you for a function whose derivative is $\sec x$, you'll have the answer $\ln\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)$ at hand.) A final note: although my answer $\sec x$ was very tidy, you can say that it results from the simplifcation of a more complicated expression. In order to get full credit for this problem, you do not need to simplify.

Part (c) was intended as an implicit differentiation problem, but you can write $y = \pm\sqrt[4]{(7-x^4)/4}$ and do the problem without implicit differentiation. I do really insist on the "$\pm$," though: the indicated fourth root is positive, by definition, but y might be negative or positive. (If you graph this function, you'll get something that looks like an ellipse that's straining to become rectangular.) The derivative of $\sqrt[4]{(7-x^4)/4}$ is

$$\frac{1}{4}((7-x^4)/4)^{-3/4}\cdot\frac{d}{dx}((7-x^4)/4) = \frac{1}{4}((7-x^4)/4)^{-3/4}\cdot(-x^3),$$

which we can rewrite as

$$\frac{-x^3}{4(\sqrt[4]{(7-x^4)/4})^3}.$$

Hence the derivative of y with respect to x is

$$\frac{-x^3}{4y^3}.$$

If you write "$\sqrt[4]{(7-x^4)/4}$" in place of y, you are getting the sign wrong half the time. If you write "$\pm\sqrt[4]{(7-x^4)/4}$" in place of y, you are telling the truth, but not the whole truth.

Here is how to do the same problem *with* implicit differentiation: Differentiate both sides of $x^4+4y^4=7$ with respect to x to get

$$4x^3+16y^3\frac{dy}{dx}=0.$$

Then solve for $\frac{dy}{dx}$ to obtain $\frac{dy}{dx}=-\frac{x^3}{4y^3}$. This is the same answer as by the first method, but I think that the second method is easier to carry out.

2. The equation $x^2 - x - 1 = 0$ has a root between 1.5 and 2.0. Suppose we employ Newton's method to find an approximation for the root, starting with 2 as our first guess for the root. What will the method output as our second guess? What will the method then output as our third guess?

To find a solution to $f(x) = 0$ near a guess a, we take $a - \dfrac{f(a)}{f'(a)}$ as a revised guess. Since $f(x) = x^2 - x - 1$, $a - \dfrac{f(a)}{f'(a)} = \dfrac{a^2+1}{2a-1}$. (You don't need to write down this last expression, but I found it to be helpful.) The guess $a = 2$ leads to the revised guess $5/3$. Repeating the process with $5/3$ in place of 2, we obtain the second revised guess $34/21$ for the root. This is in fact very close to the actual root: The true root $(1+\sqrt{5})/2$ has decimal value $1.618\ldots$, whereas $34/21 \approx 1.619$.

3a. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Use the definition of *derivative* to express $f'(0)$ as a limit.

By definition, $f'(0)$ is the limit $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$. In this case, $f(0)$ is 0, and also $x - 0$ is x, so the limit is

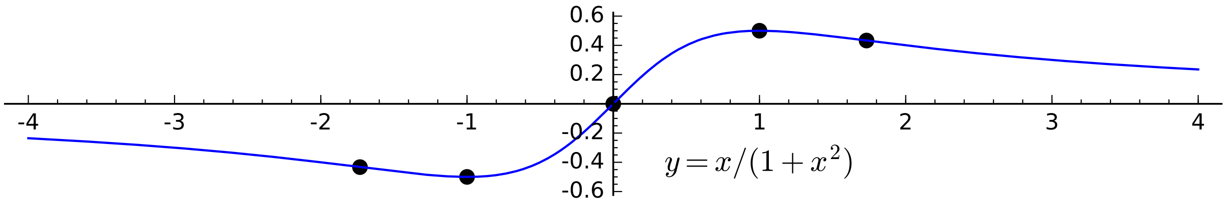
$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \left(x \sin(1/x)\right).$$

b. Decide whether or not the limit in part (a) exists; if it does, find its value.

The answer is that the limit does exist, and in fact the limit is 0. The reason is that the first factor in the product, namely x, is going to 0, while the second factor is between -1 and $+1$. Hence the product is going to 0. I don't expect people to be "proving" this rigorously, but I hope that they will provide a comprehensible explanation along these lines.

4. Sketch the graph of $y = \dfrac{x}{1+x^2}$, taking note of such features as local maxima and minima, concavity, intercepts with the axes and behavior as $x \to \pm\infty$. Be sure to include the coordinates of any points where concavity changes or where the graph has a max or min.

Here's a crude hand-drawn sketch of the curve*:



The derivative of y is $\dfrac{1-x^2}{(x^2+1)^2}$; this fraction vanishes when $x = \pm 1$; for those x-values there's a local min or a local max. The second derivative is $\dfrac{2x^3-6x}{(x^2+1)^3}$, which vanishes for $x = 0$ and for $x = \pm\sqrt{3}$. The curve is concave down from $-\infty$ to $-\sqrt{3}$, concave up from $-\sqrt{3}$ to 0, down again from 0 to $\sqrt{3}$ and finally concave up from $\sqrt{3}$ to ∞. For $x \to \infty$, y is approaching 0 from above; for $x \to -\infty$, y is approaching 0 from below.

5. Evaluate each of the following limits (if they exist):

$$\textbf{a.}\quad \lim_{x\to 0+} x^{1/x}, \qquad \textbf{b.}\quad \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right).$$

For (a), you can just reason through the problem. When x is small and positive, $1/x$ is enormous and positive. You are taking a tiny number and raising it to a huge power. The result is minuscule. This function approaches 0. l'Hôpital's rule is neither needed nor relevant.

Part (b) is a variant of a problem that I saw at SLC. The SLC problem was a lot trickier because x was approaching $-\infty$ and one needed to know that $\sqrt{x^2} = -x$ for x negative. Here, the single technique to use is to multiply and divide by $\sqrt{x^2+x+1}+x$:

$$\sqrt{x^2+x+1}-x = \left(\sqrt{x^2+x+1}-x\right)\cdot\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} = \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}.$$

Then

$$\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \frac{x+1}{\sqrt{x^2+x+1}+x} = \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}.$$

As $x \to \infty$, the quantities $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$ approach 0, so the limit is $\dfrac{1}{1+1} = \dfrac{1}{2}$.

* JK—I used `Sage`